

Induced gauge theory on a noncommutative space

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Abstract. We consider a scalar ϕ^4 theory on canonically deformed Euclidean space in 4 dimensions with an additional oscillator potential. This model is known to be renormalisable. An exterior gauge field is coupled in a gauge invariant manner to the scalar field. We extract the dynamics for the gauge field from the divergent terms of the 1-loop effective action, using a matrix basis and propose an action for the noncommutative gauge theory, which is a candidate for a renormalisable model.

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1 Introduction

Feynman rules for quantum field theory over noncommutative spaces lead for planar diagrams to the standard renormalisation problem, for non-planar ones an additional problem running under the name of infrared/ultraviolet mixing shows up.

In a previous work [1, 2], the structure of divergences is studied carefully, and it is realised that the expanded model with four marginal operators leads to a renormalisable theory:

$$S_0 = \int d^4x \left(\frac{1}{2} \phi \star [\tilde{x}_\nu, [\tilde{x}^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2} \phi \star \{ \tilde{x}^\nu, \{ \tilde{x}_\nu, \phi \}_\star \}_\star + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x). \quad (1)$$

This model fulfills the Langmann–Szabo duality [3] which motivates the added term. There are various proofs of renormalisability available [4, 5]. Similar results for fermion models have also been obtained by the Paris group [6]. We restrict ourselves to the canonical Euclidean space with constant commutation relations

$$[x^\mu \star, x^\nu] = i\theta^{\mu\nu}, \quad (2)$$

where $\theta^{ij} = -\theta^{ji} \in \mathbb{R}$, and the \star -product is given by the Weyl–Moyal product

$$f \star g(x) = e^{i\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} f(x)g(y) \Big|_{y \rightarrow x}. \quad (3)$$

The differential calculus is generated by

$$\partial_\mu f = -i [\tilde{x}_\mu, f]_\star.$$

In order to obtain the action for a gauge theory, which hopefully is renormalisable, we extract the divergent terms of the heat kernel expansion. Such a procedure leads in the commutative case to a renormalisable gauge field action. We introduce the local, unitary gauge group \mathcal{G} under which the scalar field ϕ transforms co-variantly like

$$\phi \mapsto u^\star \star \phi \star u, \quad u \in \mathcal{G}. \quad (4)$$

The approach employed here makes use of two basic ideas. First, it is well known that the \star -multiplication of a coordinate – and also of a function, of course – with a field is not a co-variant process. The product $x^\mu \star \phi$ will not transform co-variantly,

$$x^\mu \star \phi \mapsto u^\star \star x^\mu \star \phi \star u.$$

Functions of the coordinates are not effected by the gauge group. The matter field ϕ is taken to be an element of a module [7]. The introduction of co-variant coordinates

$$\tilde{X}_\nu = \tilde{x}_\nu + A_\nu \quad (5)$$

finds a remedy to this situation [8]. The gauge field A_μ and hence the co-variant coordinates transform in the following way:

$$A_\mu \mapsto iu^\star \star \partial_\mu u + u^\star \star A_\mu \star u, \quad (6)$$

$$\tilde{X}_\mu \mapsto u^\star \star \tilde{X}_\mu \star u.$$

Using co-variant coordinates we can construct an action invariant under gauge transformations. This action defines the model we are going to study in this article:

$$S = \int d^4x \left(\frac{1}{2} \phi \star [\tilde{X}_\nu, [\tilde{X}^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2} \phi \star \{ \tilde{X}^\nu, \{ \tilde{X}_\nu, \phi \}_\star \}_\star + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x). \quad (7)$$

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Secondly, we apply the heat kernel formalism. The gauge field A_μ is an external, classical gauge field coupled to ϕ . In the following sections, we will explicitly calculate the divergent terms of the one-loop effective action. In the classical case, the divergent terms determine the dynamics of the gauge field [9–11]. There have already been attempts to generalise this approach to the non-commutative realm; for non-commutative ϕ^4 theory see [12, 13]. First steps towards gauge kinetic models have been done in [14–16]. However, the results there are not completely comparable, since we have modified the free action and expand around $-\nabla^2 + \Omega^2 \tilde{x}^2$ rather than $-\nabla^2$.

A few days ago, A. de Goursac, J.-C. Wallet and R. Wulkenhaar [17] published a paper where they computed the effective action for a similar model in coordinate space. They have evaluated relevant Feynman diagrams and obtained the same results as presented here.

We note that the general formalism developed by A. Connes and A. Chamseddine [18] cannot be applied here, since in our case a tadpole contribution shows up, which is supposed to vanish in their work.

As we will see, order-by-order contributions of the employed method are not manifestly gauge invariant. But they combine in the end and provide gauge invariant results. In this paper we will discuss the case $\Omega \neq 1$ in $D = 4$ in detail, for the interesting special case $\Omega = 1$, we refer to a subsequent paper [19] and to a recent conference report [20].

In the following two sections, we describe our model and the employed method of extracting the singular contributions of the one-loop action in detail. In Sect. 4, we sketch the explicit calculations. The results are summarised in Sect. 4.5 and discussed in the final section.

2 The Model

Let us start from the action (7)

$$S = \int d^4x \left(\frac{1}{2} \phi \star [\tilde{X}_\nu, [\tilde{X}^\nu, \phi]_{\star}]_{\star} + \frac{\Omega^2}{2} \phi \star \{ \tilde{X}^\nu, \{ \tilde{X}_\nu, \phi \}_{\star} \}_{\star} + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x).$$

The expansion of S yields

$$S = S_0 + \int d^4x \left(\frac{1}{2} \phi \star (2iA^\nu \star \partial_\nu \phi - 2i\partial_\nu \phi \star A^\nu + 2(1 + \Omega^2)A_\nu \star A^\nu \star \phi - 2(1 - \Omega^2)A_\nu \star \phi \star A^\nu + 2\Omega^2 \{ \tilde{x}_\nu, (A^\nu \star \phi + \phi \star A^\nu) \}_{\star}) \right), \quad (8)$$

where S_0 denotes the free part of the action independent of A . Now we compute the second derivative:

$$\begin{aligned} \frac{\delta^2 S}{\delta \phi^2}(\psi) &= \frac{2}{\theta} H^0 \psi + \frac{\lambda}{3!} (\phi \star \phi \star \psi + \psi \star \phi \star \phi + \phi \star \psi \star \phi) \\ &+ i\partial_\nu A^\nu \star \psi - i\psi \star \partial_\nu A^\nu + 2iA^\nu \star \partial_\nu \psi - 2i\partial_\nu \psi \star A^\nu \\ &+ (1 + \Omega^2) A_\nu \star A^\nu \star \psi - 2(1 - \Omega^2) A_\nu \star \psi \star A^\nu \end{aligned}$$

$$\begin{aligned} &+ (1 + \Omega^2) \psi \star A_\nu \star A^\nu \\ &+ 2\Omega^2 (\tilde{x}_\nu \cdot (A^\nu \star \psi + \psi \star A^\nu) + (\tilde{x}_\nu \cdot \psi) \star A^\nu + A^\nu \star (\tilde{x}_\nu \cdot \psi)), \end{aligned} \quad (9)$$

where

$$H^0 = \frac{\theta}{2} \left(-\frac{\partial^2}{\partial x_\nu \partial x^\nu} + 4\Omega^2 \tilde{x}_\nu \tilde{x}^\nu + \mu^2 \right). \quad (10)$$

The oscillator term is considered as a modification of the free theory. We use the the following parametrisation of $\theta_{\mu\nu}$:

$$\begin{aligned} (\theta_{\mu\nu}) &= \begin{pmatrix} 0 & \theta & & \\ -\theta & 0 & & \\ & & 0 & \theta \\ & & -\theta & 0 \end{pmatrix}, \\ (\theta_{\mu\nu}^{-1}) &= \begin{pmatrix} 0 & -1/\theta & & \\ 1/\theta & 0 & & \\ & & 0 & -1/\theta \\ & & 1/\theta & 0 \end{pmatrix}. \end{aligned}$$

We expand the fields in the matrix base of the Moyal plane,

$$\begin{aligned} A^\nu(x) &= \sum_{p,q \in \mathbb{N}^2} A_{pq}^\nu f_{pq}(x), \\ \phi(x) &= \sum_{p,q \in \mathbb{N}^2} \phi_{pq} f_{pq}(x), \\ \psi(x) &= \sum_{p,q \in \mathbb{N}^2} \psi_{pq} f_{pq}(x). \end{aligned} \quad (11)$$

This choice of basis simplifies the calculations. In the end, we will again represent the results in the x -basis. Useful properties of this basis (which we also use in the Appendix) are reviewed in the Appendix of [1]. Using (A.2) and (A.3) from the Appendix, we obtain for (9)

$$\begin{aligned} \frac{\delta^2 S}{\delta \phi^2}(f_{mn})(x) &= \sum_{r,s \in \mathbb{N}^2} G_{rs;mn} f_{sr}(x) \\ &+ \sum_{r \in \mathbb{N}^2} \left(\frac{\lambda}{3!} \phi \star \phi + (1 + \Omega^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right)_{rm} f_{rn}(x) \\ &+ \sum_{s \in \mathbb{N}^2} \left(\frac{\lambda}{3!} \phi \star \phi + (1 + \Omega^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right)_{ns} f_{ms}(x) \\ &+ \sum_{r,s \in \mathbb{N}^2} \left(\frac{\lambda}{3!} \phi_{rm} \phi_{ns} - 2(1 - \Omega^2) A_{\nu,rm} A_{ns}^\nu \right) f_{rs}(x) \\ &+ (1 - \Omega^2) i\sqrt{\frac{2}{\theta}} \\ &\times \sum_{r \in \mathbb{N}^2} \left(\sqrt{n^1} A_{r_1 m_1}^{(1+)} f_{r_2 m_2}^{r_1 n_1 - 1} f_{r_2 n_2}^{r_1 n_1 + 1} - \sqrt{n^1 + 1} A_{r_1 m_1}^{(1-)} f_{r_2 m_2}^{r_1 n_1} f_{r_2 n_2}^{r_1 n_1 + 1} \right. \\ &\left. + \sqrt{n^2} A_{r_1 m_1}^{(2+)} f_{r_2 m_2}^{r_1 n_1} f_{r_2 n_2 - 1}^{r_1 n_1} - \sqrt{n^2 + 1} A_{r_1 m_1}^{(2-)} f_{r_2 m_2}^{r_1 n_1} f_{r_2 n_2 + 1}^{r_1 n_1} \right) \end{aligned}$$

$$\begin{aligned}
& - (1 - \Omega^2) i \sqrt{\frac{2}{\theta}} \\
& \times \sum_{s \in \mathbb{N}^2} \left(-\sqrt{m^1 + 1} A_{n^1 s^1, n^2 s^2}^{(1+)} f_{m^1 + 1 s^1, m^2 s^2} + \sqrt{m^1} A_{n^1 s^1, n^2 s^2}^{(1-)} f_{m^1 - 1 s^1, m^2 s^2} \right. \\
& \quad \left. - \sqrt{m^2 + 1} A_{n^1 s^1, n^2 s^2}^{(2+)} f_{m^1 s^1, m^2 + 1 s^2} + \sqrt{m^2} A_{n^1 s^1, n^2 s^2}^{(2-)} f_{m^1 s^1, m^2 - 1 s^2} \right), \quad (12)
\end{aligned}$$

where

$$A^{(1\pm)} = A^1 \pm iA^2, \quad A^{(2\pm)} = A^3 \pm iA^4. \quad (13)$$

We extract the (lk) -component of (12):

$$\frac{\theta}{2} \left(\frac{\delta^2 S}{\delta \phi^2} (f_{mn}) \right)_{lk} = H_{kl;mn}^0 + \frac{\theta}{2} V_{kl;mn} \equiv H_{kl;mn}, \quad (14)$$

where

$$\begin{aligned}
H_{mn;kl}^0 &= \left(\frac{\mu^2 \theta}{2} + (1 + \Omega^2)(n^1 + m^1 + 1) \right. \\
& \quad \left. + (1 + \Omega^2)(n^2 + m^2 + 1) \right) \delta_{n^1 k^1} \delta_{m^1 l^1} \delta_{n^2 k^2} \delta_{m^2 l^2} \\
& - (1 - \Omega^2) \left(\sqrt{k^1 l^1} \delta_{n^1 + 1, k^1} \delta_{m^1 + 1, l^1} \right. \\
& \quad \left. + \sqrt{m^1 n^1} \delta_{n^1 - 1, k^1} \delta_{m^1 - 1, l^1} \right) \delta_{n^2 k^2} \delta_{m^2 l^2} \\
& - (1 - \Omega^2) \left(\sqrt{k^2 l^2} \delta_{n^2 + 1, k^2} \delta_{m^2 + 1, l^2} \right. \\
& \quad \left. + \sqrt{m^2 n^2} \delta_{n^2 - 1, k^2} \delta_{m^2 - 1, l^2} \right) \delta_{n^1 k^1} \delta_{m^1 l^1} \quad (15)
\end{aligned}$$

is the field-independent part and

$$\begin{aligned}
V_{kl;mn} &= \left(\frac{\lambda}{3!} \phi \star \phi + (1 + \Omega^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right)_{lm} \delta_{nk} \\
& + \left(\frac{\lambda}{3!} \phi \star \phi + (1 + \Omega^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right)_{nk} \delta_{ml} \\
& + \left(\frac{\lambda}{3!} \phi_{lm} \phi_{nk} - 2(1 - \Omega^2) A_{\nu, lm} A_{nk}^\nu \right) \\
& + (1 - \Omega^2) i \sqrt{\frac{2}{\theta}} \\
& \times \left(\sqrt{n^1} A_{l^1 m^1, l^2 m^2}^{(1+)} \delta_{k^1 n^1 - 1, k^2 n^2} - \sqrt{n^1 + 1} A_{l^1 m^1, l^2 m^2}^{(1-)} \delta_{k^1 n^1 + 1, k^2 n^2} \right. \\
& \quad \left. + \sqrt{n^2} A_{l^1 m^1, l^2 m^2}^{(2+)} \delta_{k^1 n^1, k^2 n^2 - 1} - \sqrt{n^2 + 1} A_{l^1 m^1, l^2 m^2}^{(2-)} \delta_{k^1 n^1, k^2 n^2 + 1} \right) \\
& - (1 - \Omega^2) i \sqrt{\frac{2}{\theta}} \\
& \times \left(-\sqrt{m^1 + 1} A_{n^1 k^1, n^2 k^2}^{(1+)} \delta_{m^1 + 1 l^1, m^2 l^2} + \sqrt{m^1} A_{n^1 k^1, n^2 k^2}^{(1-)} \delta_{m^1 - 1 l^1, m^2 l^2} \right. \\
& \quad \left. - \sqrt{m^2 + 1} A_{n^1 k^1, n^2 k^2}^{(2+)} \delta_{m^1 l^1, m^2 + 1 l^2} + \sqrt{m^2} A_{n^1 k^1, n^2 k^2}^{(2-)} \delta_{m^1 l^1, m^2 - 1 l^2} \right). \quad (16)
\end{aligned}$$

The heat kernel e^{-tH^0} of the Schrödinger operator (10) can be calculated from the propagator given in [2]. In the matrix base of the Moyal plane, it has the following representation:

$$\left(e^{-tH^0} \right)_{mn;kl} = e^{-2t\sigma^2} \delta_{m+k, n+l} \prod_{i=1}^2 K_{m^i n^i; k^i l^i}(t), \quad (17)$$

$$\begin{aligned}
& K_{m, m+\alpha; l+\alpha, l}(t) \\
& = \sum_{u=0}^{\min(m, l)} \sqrt{\binom{m}{u} \binom{l}{u} \binom{\alpha+m}{m-u} \binom{\alpha+l}{l-u}} \\
& \quad \times \frac{e^{-4\Omega t (\frac{1}{2}\alpha+u)} (1 - e^{-4\Omega t})^{m+l-2u}}{\left(1 - \frac{(1-\Omega)^2}{(1+\Omega)^2} e^{-4\Omega t}\right)^{\alpha+m+l+1}} \\
& \quad \times \left(\frac{4\Omega}{(1+\Omega)^2} \right)^{\alpha+2u+1} \left(\frac{1-\Omega}{1+\Omega} \right)^{m+l-2u} \\
& = \sum_{u=0}^{\min(m, l)} \sqrt{\binom{m}{u} \binom{l}{u} \binom{\alpha+m}{m-u} \binom{\alpha+l}{l-u}} \\
& \quad \times e^{2\Omega t} \left(\frac{1-\Omega^2}{2\Omega} \sinh(2\Omega t) \right)^{m+l-2u} X_\Omega(t)^{\alpha+m+l+1} \quad (18)
\end{aligned}$$

where $2\sigma^2 = (\mu^2 \theta / 2 + 4\Omega)$, and we have defined

$$X_\Omega(t) = \frac{4\Omega}{(1+\Omega)^2 e^{2\Omega t} - (1-\Omega)^2 e^{-2\Omega t}}. \quad (20)$$

For $\Omega = 1$, the interaction part of the action simplifies a lot,

$$\begin{aligned}
V_{kl;mn} &= \left(\frac{\lambda}{3!} \phi \star \phi + 2 (\tilde{X}_\mu \star \tilde{X}^\mu - \tilde{x}^2) \right)_{lm} \delta_{nk} \\
& + \left(\frac{\lambda}{3!} \phi \star \phi + 2 (\tilde{X}_\mu \star \tilde{X}^\mu - \tilde{x}^2) \right)_{nk} \delta_{ml} + \frac{\lambda}{3!} \phi_{lm} \phi_{nk} \\
& \equiv a_{lm} \delta_{nk} + a_{nk} \delta_{ml} + \frac{\lambda}{3!} \phi_{lm} \phi_{nk}, \quad (21)
\end{aligned}$$

$$\equiv a_{lm} \delta_{nk} + a_{nk} \delta_{ml} + \frac{\lambda}{3!} \phi_{lm} \phi_{nk}, \quad (22)$$

and for the heat kernel, we obtain the following simple expression:

$$\left(e^{-tH^0} \right)_{mn;kl} = \delta_{ml} \delta_{kn} e^{-2t\sigma^2} \prod_{i=1}^2 e^{-2t(m^i + n^i)}, \quad (23)$$

$$K_{mn;kl}(t) = \delta_{ml} \prod_{i=1}^2 e^{-2t(m^i + k^i)}, \quad (24)$$

where $\sigma^2 = \frac{\mu^2 \theta}{4} + 2$.

3 Method

Given an operator P on the algebra, we write

$$Pf_{mn} = \sum_{k, l} (P_{mn})_{lk} f_{lk} = \sum_{k, l} f_{lk} P_{kl;mn}. \quad (25)$$

Then, the composition of two such operators P, Q reads

$$\begin{aligned}
 PQf_{mn} &= \sum_{k,l} (Q_{mn})_{lk} (Pf_{lk}) = \sum_{k,l,r,s} (Q_{mn})_{lk} (P_{lk})_{rs} f_{rs} \\
 &= \sum_{k,l} (Pf_{lk}) Q_{kl;mn} = \sum_{k,l,r,s} f_{rs} P_{sr;lk} Q_{kl;mn},
 \end{aligned}
 \tag{26}$$

hence

$$[PQ]_{sr;mn} = \sum_{k,l} P_{sr;lk} Q_{kl;mn}. \tag{27}$$

The trace of such an operator is then given by

$$\text{Tr}P = \sum_{m,n} P_{mn;nm}. \tag{28}$$

Bearing in mind these index rules, we can compute the regularised one-loop effective action for the model defined by the classical action (7), which is given by

$$\Gamma_{1l}^\epsilon[\phi] = -\frac{1}{2} \int_\epsilon^\infty \frac{dt}{t} \text{Tr} \left(e^{-tH} - e^{-tH^0} \right). \tag{29}$$

One way to proceed would be to use the Baker–Campbell–Hausdorff formula (it is used e.g., in [12]),

$$\begin{aligned}
 \text{Tr} \left(e^{-tH} - e^{-tH^0} \right) &= \text{Tr} \left(\left(-\frac{\theta}{2} tV + \frac{t^2}{2} \frac{\theta}{2} [H^0, V] \right. \right. \\
 &\quad \left. \left. - \frac{t^3}{6} \frac{\theta}{2} [H^0, [H^0, V]] + \frac{t^2}{2} \frac{\theta^2}{4} V^2 \right) e^{-tH^0} \right).
 \end{aligned}
 \tag{30}$$

However, for reasons of convergence, we use the Duhamel formula instead. We have to iterate the identity

$$\begin{aligned}
 e^{-tH} - e^{-tH^0} &= \int_0^t d\sigma \frac{d}{d\sigma} \left(e^{-\sigma H} e^{-(t-\sigma)H^0} \right) \\
 &= - \int_0^t d\sigma e^{-\sigma H} \frac{\theta}{2} V e^{-(t-\sigma)H^0},
 \end{aligned}
 \tag{31}$$

giving

$$\begin{aligned}
 e^{-tH} &= e^{-tH^0} - \frac{\theta}{2} \int_0^t dt_1 e^{-t_1 H^0} V e^{-(t-t_1)H^0} \\
 &+ \left(\frac{\theta}{2} \right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-t_2 H^0} V e^{-(t_1-t_2)H^0} V e^{-(t-t_1)H^0} + \dots
 \end{aligned}
 \tag{32}$$

We thus obtain¹

$$\begin{aligned}
 \Gamma_{1l}^\epsilon &= \frac{\theta}{4} \int_\epsilon^\infty dt \text{Tr} V e^{-tH^0} \\
 &\quad - \frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' t' \text{Tr} V e^{-t'H^0} V e^{-(t-t')H^0} \\
 &\quad + \frac{\theta^3}{16} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' \text{Tr} V e^{-t''H^0} \\
 &\quad \quad \times V e^{-(t'-t'')H^0} V e^{-(t-t')H^0} \\
 &\quad - \frac{\theta^4}{32} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' t''' \text{Tr} V e^{-t'''H^0} \\
 &\quad \quad \times V e^{-(t''-t''')H^0} V e^{-(t'-t'')H^0} V e^{-(t-t')H^0} \\
 &\quad + \mathcal{O}(\theta^5) \\
 &= \Gamma_{1l,1}^\epsilon[\phi] + \Gamma_{1l,2}^\epsilon[\phi] + \Gamma_{1l,3}^\epsilon[\phi] + \Gamma_{1l,4}^\epsilon[\phi] + \mathcal{O}(\theta^5).
 \end{aligned}
 \tag{33}$$

The first term in both expansions coincides. Contributions to (33) higher than fourth order are finite.

For simplicity, we introduce an additional double index notation:

$$\prod_{i=1}^2 K_{m^i n^i; k^i l^i}(t) \equiv K_{mn;kl}(t). \tag{34}$$

Indices not indexed by 1 or 2 are supposed to be double indices, unless otherwise stated.

Operators H^0 and V entering the heat kernel obey obvious scaling relations. Defining

$$\begin{aligned}
 v &= \frac{V}{1 + \Omega^2}, \\
 h^0 &= \frac{H^0}{1 + \Omega^2},
 \end{aligned}$$

and the auxiliary parameter τ

$$\tau = t (1 + \Omega^2),$$

it leads to operators depending beside on θ only on the following three parameters:

$$\rho = \frac{1 - \Omega^2}{1 + \Omega^2}, \tag{35}$$

$$\tilde{\epsilon} = \epsilon (1 + \Omega^2), \tag{36}$$

$$\tilde{\mu}^2 = \frac{\mu^2 \theta}{1 + \Omega^2}. \tag{37}$$

The task of this paper is to extract the divergent contributions of the expansion (33). In order to do so, we expand the integrands for small auxiliary parameters. The divergencies are due to infinite sums over indices occurring in the heat kernel but not in the gauge field A . After integrating over the auxiliary parameters, we obtain the divergent

¹ In the V -bilinear integral we set $t_1 - t_2 = t - t'$ so that the t_2 integration goes from 0 to t' .

contributions listed and calculated in the next section. In the end, we convert the results to x -space using

$$\sum_m B_{mm} = \frac{1}{4\pi^2\theta^2} \int d^4x B(x),$$

where $B(x) = \sum_{m,n} B_{mn} f_{mn}(x)$.

4 Calculations

We concentrate on the gauge fields and set $\lambda = 0$. The A^2 term in (16) does not need to be considered, since it leads to finite contributions in all orders. Let us examine the calculation of the Duhamel expansion (33) order by order.

4.1 First order

To first order the Duhamel expansion (33) of the effective action yields

$$\begin{aligned} \Gamma_{1l,1}^\epsilon &= \frac{\theta}{4} \int_\epsilon^\infty dt \sum_{k,l,m,n} V_{kl;mn} \left(e^{-tH^0} \right)_{nm;lk} \\ &= \frac{\theta}{4} \int_\epsilon^\infty dt e^{-2t\sigma^2} \sum_{k,l,m,n} K_{nm;lk}(t) \left\{ \left(\sqrt{n^1} A_{lm}^{(1+)} \delta_{n^1 k^1} \right)_{n^2 k^2} \right. \\ &\quad - \sqrt{n^1 + 1} A_{lm}^{(1-)} \delta_{n^1 + 1 k^1} + \sqrt{n^2} A_{lm}^{(2+)} \delta_{n^1 k^1} \\ &\quad - \sqrt{n^2 + 1} A_{lm}^{(2-)} \delta_{n^1 k^1} + \sqrt{m^1 + 1} A_{nk}^{(1+)} \delta_{l^1 m^1} \\ &\quad - \sqrt{m^1} A_{nk}^{(1-)} \delta_{l^1 m^1} + \sqrt{m^2 + 1} A_{nk}^{(2+)} \delta_{l^1 m^1} \\ &\quad \left. - \sqrt{m^2} A_{nk}^{(2-)} \delta_{l^1 m^1} \right) i \sqrt{\frac{2}{\theta}} (1 - \Omega^2) \\ &\quad + (1 + \Omega^2) \left(\left(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2 \right)_{lm} \delta_{nk} \right. \\ &\quad \left. + \left(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2 \right)_{nk} \delta_{lm} \right) \}. \end{aligned}$$

The divergences are due to partial traces of the Kernel $K(t)$, i.e., sums over indices that occur in $K(t)$, but not in the gauge fields. There are two relevant traces – no double index notation is implied here –, namely

$$\begin{aligned} &\sum_{n=0}^\infty K_{mn;nm}(t) \\ &= \sum_{n=0}^\infty \sum_{v=0}^{\min(m,n)} \binom{m}{v} \binom{n}{v} \frac{e^{-4\Omega t(\frac{1}{2}n + \frac{1}{2}m - v)} (1 - e^{-4\Omega t})^{2v}}{\left(1 - \frac{(1-\Omega)^2}{(1+\Omega)^2} e^{-4\Omega t}\right)^{n+m+1}} \\ &\quad \times \left(\frac{4\Omega}{(1+\Omega)^2} \right)^{n+m-2v+1} \left(\frac{1-\Omega}{1+\Omega} \right)^{2v} \\ &= e^{2\Omega t} X_\Omega(t)^{m+1} \left(\sum_{n=0}^\infty X_\Omega(t)^n \right) \end{aligned}$$

$$\begin{aligned} &+ m (1 - \Omega^2)^2 \frac{(e^{2\Omega t} - e^{-2\Omega t})^2}{16\Omega^2} \sum_{n=0}^\infty n X_\Omega(t)^n \Big) + \mathcal{O}(t) \\ &= e^{2\Omega t} \frac{X_\Omega(t)^{m+1}}{1 - X_\Omega(t)} \\ &\quad \times \left(1 + m \frac{(1 - \Omega^2)^2 (e^{2\Omega t} - e^{-2\Omega t})^2 X_\Omega(t)}{16\Omega^2 (1 - X_\Omega(t))} \right) + \mathcal{O}(t) \\ &= \frac{1}{(1 + \Omega^2)t} \left(1 + 2\Omega t \left(1 - \frac{(2m+1)\Omega}{1 + \Omega^2} \right) \right) + \mathcal{O}(t) \quad (38) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^\infty \sqrt{n+1} K_{m+1,n+1;n,m}(t) &= \frac{\sqrt{m+1}}{t} \frac{1 - \Omega^2}{(1 + \Omega^2)^2} \\ &\quad \times \left(1 + 2\Omega t \left(1 - \frac{2(m+1)\Omega}{1 + \Omega^2} \right) \right) + \mathcal{O}(t). \quad (39) \end{aligned}$$

The partial traces together with (A.3) and (A.6) and the identity

$$\{\tilde{x}^\mu, A_\mu\}_\star = 2\tilde{x}^\mu A_\mu = 2(\tilde{x}A)$$

lead to the following result:

$$\begin{aligned} \Gamma_{1l,1}^\epsilon &= \frac{-1}{12\pi^2} \int d^4x \left\{ \frac{-1}{\epsilon} \left(\frac{3}{2\theta} A_\nu \star A^\nu + \frac{3(1-\rho^2)}{2\theta} (2\tilde{x}A) \right) \right. \\ &\quad - \frac{\tilde{\mu}^2}{\theta} \ln \epsilon \left(\frac{3}{4} A_\nu \star A^\nu + \frac{3(1-\rho^2)}{4} (2\tilde{x}A) \right) \\ &\quad - \ln \epsilon \frac{3(1-\rho^2)}{4} \tilde{x}^2 (A_\mu \star A^\mu) + \ln \epsilon \frac{3\rho^2(1-\rho^2)}{4\theta} (2\tilde{x}A) \\ &\quad \left. - \ln \epsilon \frac{3(1-\rho^2)^2}{2} \tilde{x}^2 (\tilde{x}A) \right\} \\ &\quad + \mathcal{O}(\epsilon^0). \quad (40) \end{aligned}$$

Both, logarithmic and quadratic divergences occur. Some logarithmic divergences also stem from the constant term in the expansion of the partial traces (38) and (39), which will be called *subleading divergences*.

4.2 Second order

The second-order calculations are quite involved, and there are numerous contributions. Thus, for the clarity of presentation, we divide the calculations up. According to (33), we need to calculate

$$\begin{aligned} \Gamma_{1l,2}^\epsilon &= -\frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' \\ &\quad \times \sum_{k,l,m,n,a,b,c,d} V_{kl;mn} K(t')_{nm;ab} V_{ba;cd} K(t-t')_{dc;lk}, \quad (41) \end{aligned}$$

where the potential V is given by (16). Let us rewrite (16) in a schematic way:

$$V \text{ " = " } \tilde{X}^2 + A, \quad (42)$$

where the part “ A ” consists of two different blocks, later on referred to as first and second block. To the second order, we have to consider two potentials. Therefore, there are three different contributions which all produce divergent terms.

4.2.1 $\tilde{X}^2 - \tilde{x}^2$

First, we insert for both potentials the terms proportional to $\tilde{X} \star \tilde{X} - \tilde{x}^2$. We obtain

$$\begin{aligned}
\Gamma_{1l,2.1}^\epsilon &= -\frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' K(t')_{nm;mn} \\
&\quad \times K(t-t')_{kl;lk} (1+\Omega^2)^2 \left((\tilde{X}^{\star 2} - \tilde{x}^2)_{lm} \delta_{nk} \right. \\
&\quad \left. + (\tilde{X}^{\star 2} - \tilde{x}^2)_{nk} \delta_{lm} \right) \left((\tilde{X}^{\star 2} - \tilde{x}^2)_{ml} \delta_{kn} \right. \\
&\quad \left. + (\tilde{X}^{\star 2} - \tilde{x}^2)_{kn} \delta_{ml} \right) + \mathcal{O}(\epsilon^0) \\
&= -2(1+\Omega^2)^2 \frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' K(t')_{nm;mn} \\
&\quad \times K(t-t')_{kl;lk} (\tilde{X}^{\star 2} - \tilde{x}^2)_{lm} (\tilde{X}^{\star 2} - \tilde{x}^2)_{ml} + \mathcal{O}(\epsilon^0) \\
&= -(1+\Omega^2)^2 \frac{\theta^2}{4} \frac{1}{4\pi^2 \theta^2} \int d^4 x \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' \\
&\quad \times \frac{1}{t^2 (1+\Omega^2)^2} (\tilde{X}^{\star 2} - \tilde{x}^2)^{\star 2} + \mathcal{O}(\epsilon^0) \\
&= \frac{\ln \epsilon}{4\pi^2} \int d^4 x \frac{1}{8} (\tilde{X}^{\star 2} - \tilde{x}^2)^{\star 2} + \mathcal{O}(\epsilon^0) \\
&= \frac{\ln \epsilon}{12\pi^2 (1+\Omega^2)^2} \int d^4 x \frac{3}{8} (1+\Omega^2)^2 \left\{ (\tilde{X}^{\star 2})^{\star 2} - (\tilde{x}^2)^2 \right. \\
&\quad \left. - 2\tilde{x}^2 (A_\mu \star A^\mu) - 4\tilde{x}^2 (\tilde{x}A) \right\} + \mathcal{O}(\epsilon^0), \quad (43)
\end{aligned}$$

where we have used (38). Summation over all indices is implied.

4.2.2 $A - \tilde{X}^2$

Let us examine the field content $A^{(-1)} - \tilde{X}^2$, where A is taken from the first block. For this contribution to the effective action is given by the following expression:

$$\begin{aligned}
\Gamma_{1l,2.2}^\epsilon &= \frac{i\theta^2}{8} \sqrt{\frac{2}{\theta}} (1-\Omega^2) (1+\Omega^2) \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' \\
&\quad \times \sqrt{n^1+1} \delta_{n^1+1, k^1} A_{lm}^{(1-)} \\
&\quad \times \left((B^2 - \tilde{x}^2)_{ac} \delta_{db} + (\tilde{X}^{\star 2} - \tilde{x}^2)_{db} \delta_{ac} \right) \\
&\quad \times K_{nm;ab}(t') K_{dc;lk}(t-t') \delta_{n+a, m+b} \delta_{d+l, c+k} \\
&= \frac{i\theta^2}{8} \sqrt{\frac{2}{\theta}} (1-\Omega^2) (1+\Omega^2) \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' \\
&\quad \times \left(\sqrt{n^1+1} A_{lm}^{(1-)} (\tilde{X}^{\star 2} - \tilde{x}^2)_{ac} K_{nm;ad}(t') \right. \\
&\quad \left. \times K_{d^1 c^1 l^1 n^1+1} (t-t') \delta_{n+a, m+d} \delta_{d^1+l^1, c^1+n^1+1} \right. \\
&\quad \left. \delta_{d^2+l^2, c^2+n^2} \right) \quad (44)
\end{aligned}$$

$$\begin{aligned}
&+ \sqrt{n^1+1} A_{lm}^{(1-)} (\tilde{X}^{\star 2} - \tilde{x}^2)_{db} K_{nm;ab}(t') \\
&\quad \times K_{d^1 a^1 l^1 n^1+1} (t-t') \delta_{n+a, m+b} \delta_{d^1+l^1, a^1+n^1+1} \delta_{d^2+l^2, a^2+n^2} \Big). \quad (45)
\end{aligned}$$

The contribution of line (45) is finite, and line (44) gives two divergent contributions:

$$\begin{aligned}
\Gamma_{1l,2.2}^\epsilon &= \frac{i\theta^2}{8} \sqrt{\frac{2}{\theta}} (1-\Omega^2) (1+\Omega^2) \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' \\
&\quad \times \left(\sqrt{n^1+1} A_{cm}^{(1-)} (\tilde{X}^{\star 2} - \tilde{x}^2)_{m^1+1, c^1} \right. \\
&\quad \times K_{n^1 m^1 m^1+1, n^1+1} (t') K_{n^1+1, c^1 c^1 n^1+1} (t-t') \\
&\quad \left. + \sqrt{n^1+1} A_{cm}^{(1-)} (\tilde{X}^{\star 2} - \tilde{x}^2)_{c^1+1, m^1} K_{nm;mn}(t') \right. \\
&\quad \left. \times K_{n^1 c^1 c^1+1, n^1+1} (t-t') \right) + \mathcal{O}(\epsilon^0). \quad (46)
\end{aligned}$$

The formulae for the partial traces over two kernels are given in the Appendix, (A.4) and (A.5), resp. Only the leading terms in the expansions are necessary, since the subleading terms are already finite.

$$\begin{aligned}
\Gamma_{1l,2.2}^\epsilon &= \frac{i\theta^2}{8} \sqrt{\frac{2}{\theta}} (1-\Omega^2) (1+\Omega^2) \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' t' \\
&\quad \times \left(\sqrt{m^1+1} A_{cm}^{(1-)} (B^2 - \tilde{x}^2)_{m^1+1, c^1} \frac{1}{t(1+\Omega^2)} \frac{1-\Omega^2}{t^2(1+\Omega^2)^2} t' \right. \\
&\quad \left. + \sqrt{m^1+1} A_{c^1+1, m^1}^{(1-)} (B^2 - \tilde{x}^2)_{mc} \frac{1}{t(1+\Omega^2)} \right. \\
&\quad \left. \times \frac{1-\Omega^2}{t^2(1+\Omega^2)^2} (t-t') \right) + \mathcal{O}(\epsilon^0) \\
&= -i \frac{\theta^2}{24} \sqrt{\frac{2}{\theta}} \rho^2 \frac{\ln \epsilon}{4} \sqrt{m^1+1} \\
&\quad \times \left(2A_{cm}^{(1-)} A_{m^1+1, a^1}^{(1+)} A_{ac}^{(1-)} + 2A_{cm}^{(1-)} A_{m^1+1, a^1}^{(1-)} A_{ac}^{(1+)} \right. \\
&\quad \left. + A_{m^1+1, c^1}^{(1-)} A_{ca}^{(1-)} A_{am}^{(1+)} + A_{m^1+1, c^1}^{(1-)} A_{ca}^{(1+)} A_{am}^{(1-)} \right. \\
&\quad \left. + 4A_{cm}^{(1-)} (2\tilde{x}A)_{m^1+1, c^1} + 2A_{m^1+1, c^1}^{(1-)} (2\tilde{x}A)_{cm} \right) + \mathcal{O}(\epsilon^0). \quad (47)
\end{aligned}$$

We also need to consider the configuration $\tilde{X}^2 - A^{(-1)}$, where $A^{(-1)}$ is on the second position. The result is similar

to the one above, we obtain

$$\begin{aligned} \Gamma_{1l,2,3}^\epsilon &= \\ &-i \frac{\theta^2}{24} \sqrt{\frac{2}{\theta}} \rho^2 \frac{\ln \epsilon}{4} \sqrt{m^1+1} \\ &\times \left(A_{cm}^{(1-)} A_{m^1+1a^1}^{(1+)} A_{ac}^{(1-)} + A_{cm}^{(1-)} A_{m^1+1a^1}^{(1-)} A_{ac}^{(1+)} \right. \\ &+ 2A_{m^1+1c^1}^{(1-)} A_{ca}^{(1+)} A_{am}^{(1+)} + 2A_{m^1+1c^1}^{(1-)} A_{ca}^{(1+)} A_{am}^{(1-)} \\ &\left. + 2A_{cm}^{(1-)} (2\tilde{x}A)_{m^1+1c^1} + 4A_{m^1+1c^1}^{(1-)} (2\tilde{x}A)_{cm} \right) + \mathcal{O}(\epsilon^0) \end{aligned}$$

From the second block, we obtain the same results as above. Using (A.7) and (A.12) and taking into account the contributions from the second oscillator, we obtain

$$\begin{aligned} \Gamma_{1l,2,4}^\epsilon &= \frac{-\ln \epsilon}{12\pi^2 (1+\Omega^2)^2} \int_\epsilon^\infty d^4x (1-\Omega^2)^2 \\ &\times \left(\frac{3}{4} A_\mu \star A^\mu \star \{\tilde{x}_\nu, \star A^\nu\}_\star + \frac{3}{4} \{\tilde{x}_\nu, A^\nu\}_\star \star \{\tilde{x}_\mu, A^\mu\}_\star \right). \end{aligned} \quad (48)$$

4.2.3 $A-A$

Divergent contributions are built from fields $A^{(1+)}$ and $A^{(1-)}$, resp., $A^{(2+)}$ and $A^{(2-)}$ from the same block. Plus and minus need not to be saturated. Mixed contributions containing fields from both oscillators are finite.

$A^{(1+)} - A^{(1+)}$. Let us consider contributions with the field content $A^{(1+)} - A^{(1+)}$, from the first block. From (41) we obtain

$$\begin{aligned} \Gamma_{1l,2,5}^\epsilon &= \\ &= \frac{\theta}{4} \int_\epsilon^\infty \frac{dt}{t} e^{-2t\sigma^2} \int_0^t dt' t' (1-\Omega^2)^2 \sqrt{n^1} \sqrt{d^1} A_{lm}^{(1+)} A_{ac}^{(1+)} \\ &\times K_{nm;ab}(t') K_{dc;lk}(t-t') \delta_{k^1+1n^1} \delta_{b^1+1d^1} \delta_{n+a,m+b} \delta_{d+l,c+k} \\ &= \frac{\theta}{4} \int_\epsilon^\infty \frac{dt}{t} e^{-2t\sigma^2} \int_0^t dt' t' (1-\Omega^2)^2 \sqrt{k^1+1} \sqrt{d^1+1} \\ &\times A_{lm}^{(1+)} A_{ac}^{(1+)} K_{k^1+1, m^1, a^1, b^1} (t') K_{b^1+1, c^1, l^1, k^1} (t-t') \\ &= \frac{\theta}{4} \int_\epsilon^\infty \frac{dt}{t} e^{-2t\sigma^2} \int_0^t dt' t' (1-\Omega^2)^2 \\ &\times \left\{ (k^1+1) A_{l^1, m^1+1}^{(1+)} A_{m^1, l^1+1}^{(1+)} K_{k^1+1, m^1+1, m^1, k^1} (t') \right. \\ &\times K_{k^1+1, l^1+1, l^1, k^1} (t-t') + \sqrt{k^1+1} \sqrt{k^1+2} A_{lm}^{(1+)} \\ &\times A_{m^1, l^1+2}^{(1+)} K_{k^1+1, m^1, m^1, k^1+1} (t') K_{k^1+2, l^1+2, l^1, k^1} (t-t') \\ &\left. + \sqrt{k^1+1} \sqrt{k^1+2} A_{m^1, l^1+2}^{(1+)} A_{lm}^{(1+)} K_{k^1+2, l^1+2, l^1, k^1} (t') \right\} \end{aligned} \quad (49)$$

$$\times K_{k^1+1, m^1, m^1, k^1+1} (t-t') \left. \right\} + \mathcal{O}(\epsilon^0).$$

The other contractions of the two kernels yield finite contributions. In the other cases, the difference in indices is two or bigger, and we have for small t

$$\begin{aligned} K_{m+\beta, n+\beta; n, m}(t) &= \\ &= \sum_v \left(\binom{m+\beta}{v+\beta} \binom{m}{v} \binom{n+\beta}{v+\beta} \binom{n}{v} \right)^{1/2} e^{2\Omega t} \\ &\times \left(\frac{1-\Omega^2}{2\Omega} \sinh(2\Omega t) \right)^{2v+\beta} X_\Omega(t)^{m+n+\beta+1} \\ &= \left(\binom{m+\beta}{\beta} \binom{n+\beta}{\beta} \right)^{1/2} (1-\Omega^2)^{\beta} t^\beta X_\Omega(t)^{m+n} (1+\mathcal{O}(t)), \end{aligned}$$

with $\beta \in \mathbb{N}$ fixed. With increasing β the results are less divergent, for $\beta = 3$ already finite. We get

$$\begin{aligned} \Gamma_{1l,2,5}^\epsilon &= \\ &- \frac{\theta}{24} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{m^1+1} \sqrt{c^1+1} A_{c^1, m^1+1}^{(1+)} A_{m^1, c^1+1}^{(1+)} \\ &- \frac{\theta}{12} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{c^1} \sqrt{c^1+1} A_{c^1-1, m^1}^{(1+)} A_{m^1, c^1+1}^{(1+)} + \mathcal{O}(\epsilon^0). \end{aligned} \quad (50)$$

The same contribution comes from the second block. Therefore, there is an overall factor of 2.

$A^{(1-)} - A^{(1-)}$.

Similarly, for $A^{(1-)} - A^{(1-)}$ – first block – we obtain:

$$\begin{aligned} \Gamma_{1l,2,6}^\epsilon &= \\ &= \frac{\theta}{4} \int_\epsilon^\infty \frac{dt}{t} e^{-2t\sigma^2} \int_0^t dt' t' (1-\Omega^2)^2 \sqrt{n^1} \sqrt{d^1} A_{lm}^{(1-)} A_{ac}^{(1-)} \\ &\times K_{nm;ab}(t') K_{dc;lk}(t-t') \delta_{k^1+1n^1} \delta_{b^1+1d^1} \delta_{n+a,m+b} \delta_{d+l,c+k} \\ &= - \frac{\theta}{24} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{m^1+1} \sqrt{c^1+1} A_{c^1+1, m^1}^{(1-)} A_{m^1+1, c^1}^{(1-)} \\ &- \frac{\theta}{12} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{c^1} \sqrt{c^1+1} A_{c^1+1, m^1}^{(1-)} A_{m^1, c^1-1}^{(1-)} \\ &+ \mathcal{O}(\epsilon^0). \end{aligned} \quad (51)$$

Again, we have to take into account an overall factor of 2, which results from the equal contribution from the second block.

$A^{(1+)} - A^{(1-)}$. Next, let us consider the contribution $A^{(1+)} - A^{(1-)}$ from the first block:

$$\begin{aligned} \Gamma_{1l,2,7}^\epsilon &= - \frac{\theta}{4} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' t' e^{-2t\sigma^2} (1-\Omega^2)^2 \sqrt{n^1} \sqrt{d^1+1} \\ &\times A_{lm}^{(1+)} A_{ac}^{(1-)} K_{nm;ab}(t') K_{dc;lk}(t-t') \\ &\times \delta_{k^1+1n^1} \delta_{b^1+1d^1} \delta_{n+a,m+b} \delta_{d+l,c+k} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\theta}{4} \int_{\epsilon}^{\infty} \frac{dt}{t} \int_0^t dt' t' e^{-2t\sigma^2} (1 - \Omega^2)^2 \\
 &\times \left\{ (k^1 + 1) A_{cm}^{(1+)} A_{mc}^{(1-)} \right. \\
 &\times K_{k^2, m^2; m^2, k^2}^{k^1+1, m^1, m^1, k^1+1}(t') K_{k^2 c^2; c^2 k^2}^{k^1 c^1, c^1 k^1}(t-t') \quad (52) \\
 &+ \sqrt{(k^1 + 1)(k^1 + 2)} A_{cm}^{(1+)} A_{mc}^{(1-)} \\
 &\times K_{k^2, m^2; m^2, k^2}^{k^1+1, m^1, m^1+1, k^1+2}(t') K_{k^2 c^2; c^2 k^2}^{k^1+1 c^1+1, c^1 k^1}(t-t') \quad (53) \\
 &+ \sqrt{(k^1 + 1)(k^1 + 2)} A_{c^2 m^2}^{c^1+1 m^1+1} A_{mc}^{(1-)} \\
 &\times K_{k^2, m^2; m^2, k^2}^{k^1+2, m^1+1, m^1, k^1+1}(t') K_{k^2 c^2; c^2 k^2}^{k^1 c^1, c^1+1 k^1+1}(t-t') \quad (54) \\
 &+ (k^1 + 1) A_{c^2+1 m^2+1}^{c^1, m^1} A_{mc}^{(1-)} \\
 &\times K_{k^2+1, m^2+1; m^2, k^2}^{k^1+1, m^1, m^1, k^1+1}(t') K_{k^2 c^2; c^2+1, k^2+1}^{k^1 c^1, c^1}(t-t') \quad (55) \\
 &+ (k^1 + 1) A_{cm}^{(1+)} A_{m^1 c^1}^{(1-)} \\
 &\times K_{k^2+1, m^2+1; m^2, k^2}^{k^1+1, m^1, m^1, k^1+1}(t') K_{k^2+1, c^2+1; c^2, k^2}^{k^1, c^1, c^1, k^1}(t-t') \quad (56) \\
 &\left. + \mathcal{O}(\epsilon^0) \right\} .
 \end{aligned}$$

The contribution from the choice $A^{(1-)} - A^{(1+)}$, first block has a similar form:

$$\begin{aligned}
 \Gamma_{1l,2,8}^{\epsilon} &= -\frac{\theta}{4} \int_{\epsilon}^{\infty} \frac{dt}{t} \int_0^t dt' t' e^{-2t\sigma^2} (1 - \Omega^2)^2 \\
 &\times \left\{ (k^1 + 1) A_{cm}^{(1-)} A_{mc}^{(1+)} \right. \\
 &\times K_{k^2, m^2; m^2, k^2}^{k^1, m^1, m^1, k^1}(t') K_{k^2 c^2; c^2 k^2}^{k^1+1 c^1, c^1 k^1+1}(t-t') \quad (57) \\
 &+ \sqrt{(k^1 + 1)(k^1 + 2)} A_{cm}^{(1-)} A_{mc}^{(1+)} \\
 &\times K_{k^2, m^2; m^2, k^2}^{k^1, m^1, m^1+1, k^1+1}(t') K_{k^2 c^2; c^2 k^2}^{k^1+2 c^1+1, c^1 k^1+1}(t-t') \quad (58) \\
 &+ \sqrt{(k^1 + 1)(k^1 + 2)} A_{c^2 m^2}^{c^1+1 m^1+1} A_{mc}^{(1+)} \\
 &\times K_{k^2+1, m^2+1; m^2, k^2}^{k^1+1, m^1+1, m^1, k^1}(t') K_{k^2 c^2; c^2 k^2}^{k^1+1 c^1, c^1+1 k^1+2}(t-t') \quad (59) \\
 &+ (k^1 + 1) A_{c^2+1 m^2+1}^{c^1, m^1} A_{mc}^{(1+)} \\
 &\times K_{k^2+1, m^2+1; m^2, k^2}^{k^1, m^1, m^1, k^1}(t') K_{k^2 c^2; c^2+1, k^2+1}^{k^1+1 c^1, c^1, k^1+1}(t-t') \quad (60) \\
 &\left. + (k^1 + 1) A_{cm}^{(1-)} A_{m^1 c^1}^{(1+)} \right. \\
 &\left. + \mathcal{O}(\epsilon^0) \right\} .
 \end{aligned}$$

In order to calculate the contractions (52)–(57), we distinguish between leading and subleading contributions. Leading contributions stem only from the leading terms of the infinite sums (A.4) and (A.5). In case of quadratic divergent contributions, the subleading terms will be logarithmic divergent and need to be considered. The contractions (52) and (57) allow for these subleading divergences.

Let us first consider the leading order contributions. We get the following results:

– First contraction, term (52)

$$\begin{aligned}
 \Gamma &= -\frac{\theta}{4} \int \frac{dt}{t} e^{-2t\sigma^2} \int dt' t' (1 - \Omega^2)^2 \\
 &\times \left\{ A_{nk}^{(1+)} A_{kn}^{(1-)} + A_{nk}^{(2+)} A_{kn}^{(2-)} \right\} \\
 &\times \frac{1}{t^2 (1 + \Omega^2)^2} \frac{1}{t (1 + \Omega^2)} + \mathcal{O}(\epsilon^0) \\
 &= -\frac{\theta (1 - \Omega^2)^2}{8 (1 + \Omega^2)^3} \int \frac{dt}{t^2} e^{-2t\sigma^2} \\
 &\times \frac{1}{4\pi^2 \theta^2} \int d^4 x A_{\nu} \star A_{\nu} \Big|_{(1+), (1-)} + \mathcal{O}(\epsilon^0) \\
 &= -\frac{1}{32\pi^2 \theta} \frac{(1 - \Omega^2)^2}{(1 + \Omega^2)^3} \int d^4 x \left(\frac{1}{\epsilon} + \frac{\mu^2 \theta}{2} \ln \epsilon + 4\Omega \ln \epsilon \right) \\
 &\times A_{\nu} \star A_{\nu} \Big|_{(1+), (1-)} + \mathcal{O}(\epsilon^0) \\
 &= -\frac{1}{32\pi^2 \theta} \rho^2 \int d^4 x \left(\frac{1}{\tilde{\epsilon}} + \frac{\tilde{\mu}^2}{2} \ln \epsilon + \frac{4\Omega}{1 + \Omega^2} \ln \epsilon \right) \\
 &\times A_{\nu} \star A_{\nu} \Big|_{(1+), (1-)} + \mathcal{O}(\epsilon^0) \quad (62)
 \end{aligned}$$

– Second contraction, term (53)

$$\Gamma = \frac{\theta}{24} \rho^4 \ln \epsilon \sqrt{(m^1 + 1)(c^1 + 1)} A_{cm}^{(1+)} A_{m^1+1 c^1+1}^{(1-)} + \mathcal{O}(\epsilon^0) \quad (63)$$

– Contraction (54)

$$\begin{aligned}
 \Gamma_{1l,2,9}^{\epsilon} &= \\
 &\frac{\theta}{24} \rho^4 \ln \epsilon \sqrt{(m^1 + 1)(c^1 + 1)} A_{c^2 m^2}^{c^1+1 m^1+1} A_{mc}^{(1-)} + \mathcal{O}(\epsilon^0) \quad (64)
 \end{aligned}$$

– Contraction (55)

$$\Gamma = \frac{\theta}{48} \rho^4 \ln \epsilon \sqrt{(m^2 + 1)(c^2 + 1)} A_{c^1 m^1}^{(1+)} A_{m^1 c^1}^{(1-)} + \mathcal{O}(\epsilon^0) \quad (65)$$

– Contraction (56)

$$\Gamma = \frac{\theta}{48} \rho^4 \ln \epsilon \sqrt{(m^2 + 1)(c^2 + 1)} A_{cm}^{(1+)} A_{m^1 c^1}^{(1-)} + \mathcal{O}(\epsilon^0) \quad (66)$$

Since

$$\frac{1}{4\pi^2\theta^2} \int d^4x A_\mu \star A^\mu = \sum_{m,c} \left(A_{cm}^{(1+)} A_{mc}^{(1-)} + A_{cm}^{(2+)} A_{mc}^{(2-)} \right),$$

by $|_{(1+),(1-)}$ we denote the restriction of expressions to the fields $A^{(1+)}$ and $A^{(1-)}$. We have used this notation e.g., in (62). The missing parts are due to the field content $A^{(2+)} - A^{(2-)}$ in (41). They complement each other.

Sticking the above contributions together yields

$$\begin{aligned} \Gamma_{1l,2.9}^\epsilon &= \\ &\frac{-1}{48\pi^2}\rho^2 \left(\frac{3}{2\bar{\epsilon}\theta} + \frac{3\bar{\mu}^2}{4\theta} \ln \epsilon + \frac{6\Omega}{1+\Omega^2} \ln \epsilon \right) \int d^4x A_\mu \star A^\mu \\ &+ \frac{\theta}{24}\rho^4 \ln \epsilon \sqrt{(m^1+1)(c^1+1)} \\ &\times \left(A_{cm}^{(1+)} A_{m^1+c^1+1}^{(1-)} + A_{c^1+m^1+1}^{(1+)} A_{mc}^{(1-)} \right) \\ &+ \frac{\theta}{48}\rho^4 \ln \epsilon \sqrt{(m^2+1)(c^2+1)} \\ &\times \left(A_{cm}^{(1+)} A_{m^2+c^2+1}^{(1-)} + A_{c^2+m^2+1}^{(1+)} A_{mc}^{(1-)} \right) \\ &+ \mathcal{O}(\epsilon^0). \end{aligned} \tag{67}$$

From contractions (57)–(61) (i.e., field configuration $A^{(1-)} - A^{(1+)}$), we obtain the same result as in (67) (i.e., field configuration $A^{(1+)} - A^{(1-)}$). The second block also gives the same contributions. Therefore, we obtain an overall factor of 4.

Next, we have to examine the subleading contributions. We have to start at the sum of (52) and (57), where we want to extract the subleading divergences:

$$\begin{aligned} \Gamma_{1l,2.10}^\epsilon &= \\ &-\frac{\theta}{4} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' t' e^{-2t\sigma^2} (1-\Omega^2)^2 (k^1+1) \\ &\times \left\{ A_{cm}^{(1+)} A_{mc}^{(1-)} K_{k^1+1, m^1, m^1, k^1+1}^{k^1, c^1, c^1, k^1}(t') K_{k^2, c^2, c^2, k^2}^{k^1, c^1, c^1, k^1}(t-t') \right. \\ &\left. + A_{cm}^{(1-)} A_{mc}^{(1+)} K_{k^1, m^1, m^1, k^1}^{k^2, m^2, m^2, k^2}(t') K_{k^2+1, c^2, c^2, k^2+1}^{k^1+1, c^1, c^1, k^1+1}(t-t') \right\}. \end{aligned}$$

Therefore, we expand the partial traces. For example, the second one yields:

$$\begin{aligned} &\sum_{k^1=0}^\infty (k^1+1) K_{k^1 m^1; m^1 k^1}^{k^1, c^1, c^1, k^1}(t') K_{k^1+1, c^1; c^1, k^1+1}^{k^1+1, c^1, c^1, k^1+1}(t-t') \\ &\times \sum_{k^2=0}^\infty K_{k^2 m^2; m^2 k^2}^{k^2, m^2, m^2, k^2}(t') K_{k^2+1, c^2; c^2, k^2+1}^{k^2+1, c^2, c^2, k^2+1}(t-t') \end{aligned}$$

$$\begin{aligned} &= X_\Omega(t')^{m^1+m^2+2} X_\Omega(t-t')^{c^1+c^2+3} e^{4\Omega t} \\ &\times \sum_{k^1, k^2} (k^1+1) (X_\Omega(t') X_\Omega(t-t'))^{k^1+k^2} \\ &\times \sum_{u=0}^{\min(m^1, k^1)} \binom{m^1}{u} \binom{k^1}{u} e^{4\Omega t' u} \left(\frac{1-\Omega^2}{4\Omega} (1-e^{-4\Omega t'}) \right)^{2u} \\ &\times \sum_{v=0}^{\min(k^1+1, c^1)} \binom{k^1+1}{v} \binom{c^1}{v} e^{4\Omega(t-t')v} \\ &\times \left(\frac{1-\Omega^2}{4\Omega} (1-e^{-4\Omega(t-t')}) \right)^{2v} \\ &\times \sum_{r=0}^{\min(m^2, k^2)} \binom{m^2}{r} \binom{k^2}{r} e^{4\Omega t' r} \left(\frac{1-\Omega^2}{4\Omega} (1-e^{-4\Omega t'}) \right)^{2r} \\ &\times \sum_{s=0}^{\min(k^2, c^2)} \binom{k^2}{s} \binom{c^2}{s} e^{4\Omega(t-t')s} \\ &\times \left(\frac{1-\Omega^2}{4\Omega} (1-e^{-4\Omega(t-t')}) \right)^{2s} \\ &= (1+4\Omega t - (m^1+m^2+2)(1+\Omega^2)t' \\ &\quad - (c^1+c^2+3)(1+\Omega^2)(t-t')) \\ &\quad \times \frac{1}{(1-X_\Omega(t')X_\Omega(t-t'))^3} \\ &\quad + (1-\Omega^2)^2 \left(m^1 \frac{2t'^2}{t^4} + m^2 \frac{t'^2}{t^4} + c^1 \frac{2(t-t')^2}{t^4} \right. \\ &\quad \left. + c^2 \frac{(t-t')^2}{t^4} \right) + \dots \\ &= (1+4\Omega t - (m^1+m^2+2)(1+\Omega^2)t' \\ &\quad - (c^1+c^2+3)(1+\Omega^2)(t-t')) \\ &\quad \times \left(\frac{1}{(1+\Omega^2)^3 t^3} + \frac{3(1+\Omega^4)}{(1+\Omega^2)^4 t^2} - \frac{3t'(1-\Omega^2)^2}{(1+\Omega^2)^4 t^3} \right) \\ &\quad + (1-\Omega^2)^2 \left(m^1 \frac{2t'^2}{t^4} + m^2 \frac{t'^2}{t^4} + c^1 \frac{2(t-t')^2}{t^4} \right. \\ &\quad \left. + c^2 \frac{(t-t')^2}{t^4} \right) + \dots \end{aligned}$$

using $X_\Omega(t)^m = 1 - (1+\Omega^2)mt + \mathcal{O}(t^2)$ and the geometric series given in (A.1). Thus, we obtain for $\Gamma_{1l,2.10}^\epsilon$ to subleading order the following expressions:

$$\begin{aligned} \Gamma_{1l,2.10}^\epsilon &= \frac{\theta}{12}\rho^4 \ln \epsilon \left(A_{cm}^{(1+)} A_{mc}^{(1-)} (2c^1+c^2) + A_{cm}^{(1-)} A_{mc}^{(1+)} (2c^1+c^2) \right) \\ &- \frac{\ln \epsilon}{24\pi^2} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^2 \int d^4x \left\{ -\frac{6\Omega}{\theta(1+\Omega^2)} A_\mu \star A^\mu \right. \\ &\quad \left. + \frac{9}{4\theta} A_\mu \star A^\mu + \frac{3}{4} \tilde{x}^2 (A_\mu \star A^\mu) \right\} \Big|_{(1+),(1-)} \\ &- \frac{1}{16\pi^2\theta} \ln \epsilon \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^2 \int d^4x A_\mu \star A^\mu \Big|_{(1+),(1-)} + \mathcal{O}(\epsilon^0) \end{aligned}$$

$$= -\frac{\ln \epsilon}{12\pi^2 (1+\Omega^2)^2} \int d^4x \frac{(1-\Omega^2)^4}{(1+\Omega^2)^2} \times \left(-\frac{1}{4} \tilde{x}^2 (A_\mu \star A^\mu) + \frac{1}{2\theta} A_\mu \star A^\mu \right) \Big|_{(1+), (1-)} \quad (68)$$

$$+ \frac{\theta \ln \epsilon}{12 (1+\Omega^2)^2} \frac{(1-\Omega^2)^4}{(1+\Omega^2)^2} A_{cm}^{(1+)} A_{cm}^{(1-)} (c^1 + m^1) - \frac{\ln \epsilon}{24\pi^2} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^2 \int d^4x \left\{ -\frac{6\Omega}{\theta (1+\Omega^2)} A_\mu \star A^\mu + \frac{9}{4\theta} A_\mu \star A^\mu + \frac{3}{4} \tilde{x}^2 (A_\mu \star A^\mu) \right\} \Big|_{(1+), (1-)} \quad (69)$$

$$- \frac{1}{16\pi^2 \theta} \ln \epsilon \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^2 \int d^4x A_\mu \star A^\mu \Big|_{(1+), (1-)} + \mathcal{O}(\epsilon^0), \quad (70)$$

Summation of above contributions. Let us sum the contributions (50), (51), term 2 and 3 of (67) and (68) taking into account the correct multiplicities. We obtain

$$\begin{aligned} \Gamma_{11,2,11}^\epsilon &= -\frac{\ln \epsilon}{12\pi^2 (1+\Omega^2)^2} \int d^4x \frac{(1-\Omega^2)^4}{(1+\Omega^2)^2} \times \left(-\frac{1}{2} \tilde{x}^2 (A_\mu \star A^\mu) + \frac{1}{\theta} A_\mu \star A^\mu \right) \\ &+ \frac{\theta \ln \epsilon}{6 (1+\Omega^2)^2} \frac{(1-\Omega^2)^4}{(1+\Omega^2)^2} A_{nk}^{(1+)} A_{kn}^{(1-)} (n^1 + k^1) \\ &+ \frac{\theta}{6} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{(m^1+1)(c^1+1)} \\ &\times \left(A_{cm}^{(1+)} A_{m^1+1c^1+1}^{(1-)} + A_{c^1+1m^1+1}^{(1+)} A_{mc}^{(1-)} \right) \\ &+ \frac{\theta}{12} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{(m^2+1)(c^2+1)} \\ &\times \left(A_{cm}^{(1+)} A_{m^1 c^1}^{(1-)} + A_{c^1 m^1}^{(1+)} A_{mc}^{(1-)} \right) \\ &- \frac{\theta}{12} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{m^1+1} \sqrt{c^1+1} A_{c^1 m^1+1}^{(1+)} A_{m^1 c^1+1}^{(1-)} \\ &- \frac{\theta}{6} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{c^1} \sqrt{c^1+1} A_{c^1 m^1}^{(1+)} A_{m^1 c^1+1}^{(1+)} \\ &- \frac{\theta}{12} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{m^1+1} \sqrt{c^1+1} A_{c^1+1m^1}^{(1-)} A_{m^1+1c^1}^{(1-)} \\ &- \frac{\theta}{6} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \sqrt{c^1} \sqrt{c^1+1} A_{c^1+1m^1}^{(1-)} A_{m^1 c^1}^{(1-)} \\ &= -\frac{\ln \epsilon}{12\pi^2 (1+\Omega^2)^2} \int d^4x \frac{(1-\Omega^2)^4}{(1+\Omega^2)^2} \times \left(-\frac{1}{2} \tilde{x}^2 (A_\mu \star A^\mu) + \frac{1}{\theta} A_\mu \star A^\mu \right) \end{aligned} \quad (71)$$

$$\begin{aligned} &+ \frac{\ln \epsilon}{12} \theta \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \left\{ 2A_{nk}^{(1+)} A_{kn}^{(1-)} (n^1+k^1+1) - 2A_{nk}^{(1+)} A_{kn}^{(1-)} \right. \\ &+ \sqrt{(m^1+1)(c^1+1)} \left(A_{cm}^{(1+)} A_{m^1+1c^1+1}^{(1-)} + A_{c^1+1m^1+1}^{(1+)} A_{mc}^{(1-)} \right) \\ &- \sqrt{m^1+1} \sqrt{c^1+1} \left(A_{c^1 m^1+1}^{(1+)} A_{m^1 c^1+1}^{(1+)} + A_{c^1+1m^1}^{(1-)} A_{m^1+1c^1}^{(1-)} \right) \\ &- 2\sqrt{c^1} \sqrt{c^1+1} \left(A_{c^1+1m^1}^{(1+)} A_{m^1 c^1+1}^{(1+)} + A_{c^1+1m^1}^{(1-)} A_{m^1 c^1+1}^{(1-)} \right) \left. \right\} \\ &+ \frac{\theta}{12} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \ln \epsilon \left\{ \sqrt{(m^2+1)(c^2+1)} \right. \\ &\times \left(A_{c^1 m^1}^{(1+)} A_{m^1 c^1+1}^{(1-)} + A_{c^1+1m^1}^{(1+)} A_{m^1 c^1}^{(1-)} \right) \\ &+ \sqrt{(m^1+1)(c^1+1)} \\ &\times \left(A_{cm}^{(1+)} A_{m^1+1c^1+1}^{(1-)} + A_{c^1+1m^1+1}^{(1+)} A_{mc}^{(1-)} \right) \left. \right\} \\ &= -\frac{\ln \epsilon}{12\pi^2 (1+\Omega^2)^2} \int d^4x \frac{(1-\Omega^2)^4}{(1+\Omega^2)^2} \\ &\times \left(-\frac{1}{2} \tilde{x}^2 (A_\mu \star A^\mu) + \frac{3}{2\theta} A_\mu \star A^\mu \right) \Big|_{(1+), (1-)} \\ &- \frac{\ln \epsilon}{12\pi^2} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \int d^4x \left\{ -\frac{1}{4} \{ \tilde{x}_\nu, A^\nu \} \star \{ \tilde{x}_\mu, A^\mu \} \star \right. \\ &- \frac{1}{2} \tilde{x}_\nu \star \tilde{x}_\mu \star A^\nu \star A^\mu \left. \right\} \Big|_{(1+), (1-)} \\ &+ \frac{\ln \epsilon}{12} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \left\{ \frac{1}{2\pi^2} \int d^4x (\tilde{x}^2 A^\sigma) \star A_\sigma \right. \\ &- 2\theta A_{nk}^{(1+)} A_{kn}^{(1-)} (n^1+n^2+1) \left. \right\} \Big|_{(1+), (1-)} \\ &= -\frac{\ln \epsilon}{12\pi^2} \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^4 \int d^4x \left\{ -\frac{1}{2} \tilde{x}^2 (A_\mu \star A^\mu) \right. \\ &+ \frac{3}{2\theta} A_\mu \star A^\mu - \frac{1}{4} \{ \tilde{x}_\nu, A^\nu \} \star \{ \tilde{x}_\mu, A^\mu \} \star \\ &- \frac{1}{2} \tilde{x}_\nu \star \tilde{x}_\mu \star A^\nu \star A^\mu - \frac{1}{4} \tilde{x}_\nu \star A^\mu \star \tilde{x}^\nu \star A_\mu \left. \right\} \Big|_{(1+), (1-)}. \end{aligned} \quad (74)$$

We have made use of the matrix base expressions quoted in the Appendix.

4.2.4 Second order result

In the end, we can add up all the different terms. The result to the second order is found to be

$$\begin{aligned} \Gamma_{11,2}^\epsilon &= \\ &= \frac{-\rho^2 \ln \epsilon}{12\pi^2} \int d^4x \left\{ \left(-\frac{3}{8} (\tilde{X}_\mu \star \tilde{X}^\mu \star \tilde{X}_\nu \star \tilde{X}^\nu - (\tilde{x}^2)^2) \right) \right. \\ &\quad \left. + \frac{3}{4} \tilde{x}^2 (A_\mu \star A^\mu) + \frac{3}{2} \tilde{x}^2 (\tilde{x} A) \right\} \end{aligned} \quad (75)$$

$$+ \left(\frac{1}{\tilde{\epsilon}} \frac{3}{2\theta} A_\mu \star A^\mu + \frac{\tilde{\mu}^2}{\theta} \ln \epsilon \frac{3}{4} A_\mu \star A^\mu \right. \quad (76)$$

$$\left. + \ln \epsilon \frac{6\Omega}{\theta(1+\Omega^2)} A_\mu \star A^\mu - \frac{1}{4} \rho^2 A_\mu \star \tilde{x}_\nu \star A^\mu \star \tilde{x}^\nu \right)$$

$$+ \left(-\frac{6\Omega}{\theta(1+\Omega^2)} A_\mu \star A^\mu + \frac{9}{4\theta} A_\mu \star A^\mu \right. \quad (77)$$

$$+ \frac{3}{4} \tilde{x}^2 (A_\mu \star A^\mu) - \frac{3}{2\theta} A_\mu \star A^\mu$$

$$- \frac{1}{2} \rho^2 \tilde{x}^2 (A_\mu \star A^\mu) - \frac{3}{4} \rho^2 A_\mu \star A^\mu$$

$$\left. - \rho^2 \left(\frac{1}{4} \{ \tilde{x}_\nu, A^\nu \}_\star \star \{ \tilde{x}_\mu, A^\mu \}_\star + \frac{1}{2} \tilde{x}_\mu \tilde{x}_\nu (A^\mu \star A^\nu) \right) \right)$$

$$+ \left(\frac{3}{4} A_\mu \star A^\mu \star \{ \tilde{x}_\nu, A^\nu \}_\star + \frac{3}{4} \{ \tilde{x}_\nu, A^\nu \}_\star \star \{ \tilde{x}_\mu, A^\mu \}_\star \right) \quad (78)$$

$$+ \mathcal{O}(\epsilon^0).$$

4.3 Third order

In third order, the one-loop effective action is given by

$$\Gamma_{1l,3}^\epsilon = \frac{\theta^3}{16} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' e^{-2t\sigma^2} \quad (79)$$

$$\times \sum_{k,l,m,n,a,b,c,d,g,f,u,v} \delta_{n+a,m+b} \delta_{d+g,c+f} \delta_{v+l,u+k}$$

$$\times V_{kl;mn} K(t'')_{nm;ab} V_{ba;cd} K(t'-t'')_{dc;gf} V_{fg;uv} K(t-t')_{vu;lk}.$$

There are two different divergent contributions.

$A-A-\tilde{X}^2$. In order to obtain a divergent contribution both A fields have to be taken from the same block, one with index “-” and one with index “+”. So, let us consider the case $A^{(1+)}A^{(1-)}B^2$, where both $A^{(1-)}$ and $A^{(1+)}$ are taken from the first block of (16). Then, we obtain

$$\Gamma_{1l,3.1}^\epsilon = \frac{\theta^3}{16} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' e^{-2t\sigma^2} (1+\Omega^2) \quad (80)$$

$$\times (1-\Omega^2)^2 \frac{2}{\theta} \delta_{n+a,m+b} \delta_{d+g,c+f} \delta_{v+l,u+k}$$

$$\times \sqrt{n^1} A_{lm}^{(1+)} \delta_{k^1 n^1 -1} \sqrt{d^1 + 1} A_{ac}^{(1-)} \delta_{d^1 + 1 b^1}$$

$$\times K_{nm;ab}(t'') K_{dc;gf}(t'-t'') K_{vu;lk}(t-t')$$

$$\times \left((B_\mu \star B^\mu - \tilde{x}^2)_{gu} \delta_{vf} + (B_\mu \star B^\mu - \tilde{x}^2)_{vf} \delta_{gu} \right)$$

$$= \frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' e^{-2t\sigma^2} (1+\Omega^2) (1-\Omega^2)^2 \quad (81)$$

$$\times \sum_{c,d,l,m} (d^1 + 1) A_{lm}^{(1+)} A_{mc}^{(1-)} (B_\nu \star B^\nu - \tilde{x}^2)_{cl}$$

$$\times K_{d^1+1, m^1, m^1, d^1+1} (t'') K_{dc;cd}(t'-t'') K_{dl;ld}(t-t')$$

$$+ \mathcal{O}(\epsilon^0)$$

$$= \frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' e^{-2t\sigma^2} (1+\Omega^2) \quad (82)$$

$$\times (1-\Omega^2)^2 \frac{1}{t^2 (1+\Omega^2)^2} \frac{1}{t(1+\Omega^2)}$$

$$\times \sum_{c,l,m} A_{lm}^{(1+)} A_{mc}^{(1-)} (B_\nu \star B^\nu - \tilde{x}^2)_{cl} + \mathcal{O}(\epsilon^0)$$

$$= \frac{-\ln \epsilon}{12\pi^2} \frac{1}{16} \int d^4 x \left(\frac{1-\Omega^2}{1+\Omega^2} \right)^2 A^{(1+)} \star A^{(1-)} \star (B_\nu \star B^\nu - \tilde{x}^2)$$

$$+ \mathcal{O}(\epsilon^0). \quad (83)$$

There is an equal contribution coming from the second block. Therefore, we have a multiplicity factor of 12, since we can additionally rearrange the fields in the product “ AAB^2 ”. Thus, we get

$$\Gamma_{1l,3.1}^\epsilon = \frac{-\ln \epsilon}{16\pi^2} \int d^4 x \rho^2 A_\mu \star A^\mu \star (A_\nu \star A^\nu + 2\tilde{x}A) + \mathcal{O}(\epsilon^0). \quad (84)$$

$A-A-A$. All the fields have to be chosen from the same block of (16). Otherwise the contributions are finite. Either all three fields are from the same oscillator or only two of them. In the latter case the signs belonging to the same oscillator have to be saturated.

We first examine the expression related to the choice $A^{(1+)}-A^{(1-)}-A^{(1-)}$, where all the fields are taken from the first block. The calculation yields

$$\Gamma_{1l,3.2}^\epsilon = \frac{\theta^3}{16} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' e^{-2\sigma^2 t} \delta_{n+a,m+b} \quad (85)$$

$$\times \delta_{d+p,c+q} \delta_{s+l,r+k} (1-\Omega^2)^3 (-i) \left(\frac{2}{\theta} \right)^{3/2} \sqrt{n^1} A_{lm}^{(1+)}$$

$$\times \delta_{k^1 n^1 -1} \sqrt{d^1 + 1} A_{ac}^{(1-)} \delta_{d^1 + 1 b^1} \sqrt{s^1 + 1} A_{pr}^{(1-)} \delta_{s^1 + 1 q^1}$$

$$\times K(t'')_{nm;ab} K(t'-t'')_{dc;pq} K(t-t')_{sr;lk}$$

$$= -i \left(\frac{2}{\theta} \right)^{3/2} \frac{\theta^3}{16} (1-\Omega^2)^3 \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' e^{-2\sigma^2 t}$$

$$\times \sum_{m,p,r} A_{rm}^{(1+)} A_{mp}^{(1-)} A_{p^1+1, r^1}^{(1-)}$$

$$\times \sum_{n^1} (n^1 + 1)^{3/2} K(t'')_{n^1+1, m^1; m^1, n^1+1}$$

$$\times K(t'-t'')_{n^1 p^1; p^1+1, n^1+1} K(t-t')_{n^1 r^1; r^1 n^1}$$

$$\times \sum_{n^2} K(t'')_{n^2 m^2; m^2 n^2} K(t'-t'')_{n^2 p^2; p^2 n^2}$$

$$\times K(t-t')_{n^2 r^2; r^2 n^2} + \mathcal{O}(\epsilon^0)$$

$$= i \ln \epsilon \sqrt{\frac{2}{\theta}} \rho^4 \frac{\theta^2}{96} \sum_{m,p,r} \sqrt{p^1+1} A_{rm}^{(1+)} A_{mp}^{(1-)} A_{p^1+1, r^1}^{(1-)} + \mathcal{O}(\epsilon^0). \quad (86)$$

The expression for $A^{(1+)}A^{(1-)}A^{(1+)}$ is of a slightly different form:

$$\begin{aligned} \Gamma_{1l,3.3}^\epsilon &= -i \ln \epsilon \sqrt{\frac{2}{\theta}} \frac{\theta^2 (1 - \Omega^2)^4}{4(1 + \Omega^2)^4} \frac{1}{24} \\ &\times \sum_{m,p,r} \sqrt{r^1 + 1} A_{rm}^{(1+)} A_{mp}^{(1-)} A_{p^1 r^1 + 1}^{(1+)} + \mathcal{O}(\epsilon^0). \end{aligned} \quad (87)$$

Both terms, (86) and (87) appear six times. Note the difference in the overall sign. Therefore, we obtain the contribution

$$\begin{aligned} \Gamma_{1l,3.4}^\epsilon &= \Gamma_{1l,3.2}^\epsilon + \Gamma_{1l,3.3}^\epsilon \\ &= i \ln \epsilon \sqrt{\frac{2}{\theta}} \frac{\theta^2}{16} \left(\frac{1 - \Omega^2}{1 + \Omega^2} \right)^4 \\ &\times \sum_{rmp} \sqrt{p^1 + 1} \left(A_{rm}^{(1+)} A_{mp}^{(1-)} A_{p^1 r^1 + 1}^{(1-)} - A_{pm}^{(1+)} A_{mr}^{(1-)} A_{r^1 p^1 + 1}^{(1+)} \right) \\ &+ \mathcal{O}(\epsilon^0). \end{aligned} \quad (88)$$

Also the field content of the form $A^{(1+)}A^{(1-)}A^{(2\pm)}$ produces a divergent contribution, e.g.,

$$\begin{aligned} \Gamma_{1l,3.4}^\epsilon &= i \ln \epsilon \sqrt{\frac{2}{\theta}} \frac{\theta^2 (1 - \Omega^2)^4}{4(1 + \Omega^2)^4} \frac{1}{24} \\ &\times \sum_{m,p,r} \sqrt{l^2 + 1} A_{i^1 m^1}^{(1+)} A_{mc}^{(1-)} A_{cl}^{(2-)} + \mathcal{O}(\epsilon^0). \end{aligned} \quad (89)$$

Comparing (A.10) and (A.12), we see that the above expressions (plus the ones we have omitted here) are equal to

$$\begin{aligned} \Gamma_{1l,3.5}^\epsilon &= \frac{\ln \epsilon}{12\pi^2} \frac{3}{8} \left(\frac{1 - \Omega^2}{1 + \Omega^2} \right)^4 \\ &\times \int d^4x (A_\mu \star A^\mu \star \{\tilde{x}_\nu, A^\nu\}_\star - \tilde{x}_\nu \star A_\mu \star A^\nu \star A^\mu) \\ &+ \mathcal{O}(\epsilon^0). \end{aligned} \quad (90)$$

Summing up all the partial contributions, we obtain for the action in the third order:

$$\begin{aligned} \Gamma_{1l,3}^\epsilon &= \frac{-\ln \epsilon}{12\pi^2} \int d^4x \left\{ \frac{3}{4} \rho^2 \left((A_\mu \star A^\mu)^{\star 2} + A_\mu \star A^\mu \star \{\tilde{x}_\nu, \star A^\nu\}_\star \right) \right. \\ &\quad \left. - \frac{1}{2} \rho^4 (\tilde{x}_\nu \star A_\mu \star A^\nu \star A^\mu + A_\mu \star A^\mu \star \{\tilde{x}_\nu, \star A^\nu\}_\star) \right\} \\ &+ \mathcal{O}(\epsilon^0). \end{aligned} \quad (91)$$

4.4 Fourth order

The fourth order expression of the effective action reads:

$$\Gamma_{1l,4}^\epsilon = -\frac{\theta^4}{32} \int_\epsilon^\infty \frac{dt}{t} e^{-2\sigma^2 t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' t''' e^{-2\sigma^2 t} \quad (92)$$

$$\begin{aligned} &\times \sum \delta_{n+a,m+b} \delta_{d+e,c+f} \delta_{h+i,g+j} \delta_{q+l,p+k} \\ &\times V_{kl;mn} K(t''')_{nm;ab} V_{ba;cd} K(t'' - t''')_{dc,ef} \\ &\times V_{fe;gh} K(t' - t'')_{hg;ij} V_{ji;pq} K(t - t')_{qp;lk}. \end{aligned}$$

There is only one divergent contribution stemming from the field content $A - A - A - A$. All the fields have to come from the same block. Fields from the second oscillator may mix with fields from the first in a single expression, but the signs need to be saturated for each oscillator.

The explicit calculation yields the result:

$$\begin{aligned} \Gamma_{1l,4}^\epsilon &= \frac{-\ln \epsilon}{12\pi^2} \int d^4x \frac{1}{8} \rho^4 \\ &\times (-2(A_\mu \star A^\mu)^{\star 2} - A_\mu \star A_\nu \star A^\mu \star A^\nu) + \mathcal{O}(\epsilon^0). \end{aligned} \quad (93)$$

4.5 Summed up result

Summing up the order-by-order result, we end up at the final expression for the gauge field action:

$$\begin{aligned} \Gamma_{1l}^\epsilon &= \frac{1}{192\pi^2} \int d^4x \left\{ \frac{24}{\tilde{\epsilon}\theta} (1 - \rho^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right. \\ &+ \ln \epsilon \left(\frac{12}{\theta} (1 - \rho^2) (\tilde{\mu}^2 - \rho^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right. \\ &+ 6(1 - \rho^2)^2 \left((\tilde{X}_\mu \star \tilde{X}^\mu)^{\star 2} - (\tilde{x}^2)^2 \right) \\ &\quad \left. \left. - \rho^4 F_{\mu\nu} F^{\mu\nu} \right) \right\}, \end{aligned} \quad (94)$$

where the field strength is given by

$$F_{\mu\nu} = -i[\tilde{x}_\mu, A_\nu]_\star + i[\tilde{x}_\nu, A_\mu]_\star - i[A_\mu, A_\nu]_\star. \quad (95)$$

5 Conclusions

Our main result is summarised in (94): Both, the linear in ϵ and the logarithmic in ϵ divergent term turn out to be gauge invariant. The logarithmically divergent part is an interesting candidate for a renormalisable gauge interaction. We note that the resulting action has been proposed by R. Wulkenhaar and one of us (H.G.) in previous reports. As far as we know, this action did not appear before in string theory. The sign of the term quadratic in the co-variant coordinates may change depending on whether $\tilde{\mu}^2 \leq \rho^2$. This reflects a phase transition. In a forthcoming work (H.G. and H. Steinacker, in preparation), we were able to analyse in detail an action like (94) in two dimensions. The case $\Omega = 1$ ($\rho = 0$) is of course of particular

interest. One obtains a matrix model. We shall return to a study of these models in a forthcoming publication [19]. In the limit $\Omega \rightarrow 0$, we obtain just the standard deformed Yang–Mills action. Furthermore, the action (94) allows to study the limit $\theta \rightarrow \infty$.

In addition, we will attempt to study the perturbative quantisation. One of the problems of quantising action (94) is connected to the tadpole contribution, which is non-vanishing and hard to eliminate. The Paris group arrived at similar conclusions.

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Appendix

Useful geometric series and variants:

$$\begin{aligned} \sum_{n=0}^{\infty} X^n &= \frac{1}{1-X}, \\ \sum_{n=0}^{\infty} nX^n &= \frac{X}{(1-X)^2}, \\ \sum_{n=0}^{\infty} (n+1)X^n &= \frac{1}{(1-X)^2}, \\ \sum_{n=0}^{\infty} (n+1)^2 X^n &= \frac{1+X}{(1-X)^3}, \\ \sum_{n=0}^{\infty} (n+1)nX^n &= \frac{2X}{(1-X)^3}. \end{aligned} \tag{A.1}$$

Derivatives yield in the matrix basis

$$\begin{aligned} (\partial_\nu \psi)(x) &= \sum_{p,q \in \mathbb{N}^2} \psi_{pq} \partial_\nu f_{pq}(x) \\ &= \sum_{p,q \in \mathbb{N}^2} \psi_{pq} \left(\frac{\partial a^1}{\partial x^\nu} \frac{\partial f_{pq}}{\partial a^1} + \frac{\partial \bar{a}^1}{\partial x^\nu} \frac{\partial f_{pq}}{\partial \bar{a}^1} + \frac{\partial a^2}{\partial x^\nu} \frac{\partial f_{pq}}{\partial a^2} + \frac{\partial \bar{a}^2}{\partial x^\nu} \frac{\partial f_{pq}}{\partial \bar{a}^2} \right) \\ &= \frac{1}{\sqrt{2\theta}} \sum_{p,q \in \mathbb{N}^2} \left((\delta_{\nu,1} + i\delta_{\nu,2})(f_{pq} \star \bar{a}^1 - \bar{a}^1 \star f_{pq}) \right. \\ &\quad + (\delta_{\nu,1} - i\delta_{\nu,2})(a^1 \star f_{pq} - f_{pq} \star a^1) \\ &\quad + (\delta_{\nu,3} + i\delta_{\nu,4})(f_{pq} \star \bar{a}^2 - \bar{a}^2 \star f_{pq}) \\ &\quad \left. + (\delta_{\nu,3} - i\delta_{\nu,4})(a^2 \star f_{pq} - f_{pq} \star a^2) \right) \psi_{pq} \\ &= \frac{1}{\sqrt{2\theta}} \sum_{p,q \in \mathbb{N}^2} \left((\delta_{\nu,1} + i\delta_{\nu,2}) \right. \\ &\quad \left. \times \left(\sqrt{q^1} f_{p^1, q^1-1} - \sqrt{p^1+1} f_{p^1+1, q^1} \right) f_{p^2 q^2} \right. \end{aligned}$$

$$\begin{aligned} &+ (\delta_{\nu,1} - i\delta_{\nu,2}) \left(\sqrt{p^1} f_{p^1-1, q^1} - \sqrt{q^1+1} f_{p^1, q^1+1} \right) f_{p^2 q^2} \\ &+ (\delta_{\nu,3} + i\delta_{\nu,4}) f_{p^1 q^1} \left(\sqrt{q^2} f_{p^2 q^2-1} - \sqrt{p^2+1} f_{p^2+1, q^2} \right) \\ &+ (\delta_{\nu,3} - i\delta_{\nu,4}) f_{p^1 q^1} \left(\sqrt{p^2} f_{p^2-1, q^2} - \sqrt{q^2+1} f_{p^2, q^2+1} \right) \psi_{pq} \\ &= \frac{1}{\sqrt{2\theta}} \sum_{p,q \in \mathbb{N}^2} \left((\delta_{\nu,1} + i\delta_{\nu,2}) \left(\sqrt{q^1+1} \psi_{\frac{p^1 q^1+1}{p^2 q^2}} - \sqrt{p^1} \psi_{\frac{p^1-1 q^1}{p^2 q^2}} \right) \right. \\ &\quad + (\delta_{\nu,1} - i\delta_{\nu,2}) \left(\sqrt{p^1+1} \psi_{\frac{p^1+1 q^1}{p^2 q^2}} - \sqrt{q^1} \psi_{\frac{p^1 q^1-1}{p^2 q^2}} \right) \\ &\quad + (\delta_{\nu,3} + i\delta_{\nu,4}) \left(\sqrt{q^2+1} \psi_{\frac{p^1 q^1}{p^2 q^2+1}} - \sqrt{p^2} \psi_{\frac{p^1 q^1}{p^2-1 q^2}} \right) \\ &\quad \left. + (\delta_{\nu,3} - i\delta_{\nu,4}) \left(\sqrt{p^2+1} \psi_{\frac{p^1 q^1}{p^2+1 q^2}} - \sqrt{q^2} \psi_{\frac{p^1 q^1}{p^2 q^2-1}} \right) \right) f_{pq}. \end{aligned} \tag{A.2}$$

We also compute $\tilde{x}_\mu \cdot \psi$ in the matrix basis

$$\begin{aligned} 2\tilde{x}_\nu \cdot \psi(x) &= \sum_{p,q \in \mathbb{N}^2} \psi_{pq} (\delta_{\nu,1}(\theta^{-1})_{12}(x^2 \star f_{pq} + f_{pq} \star x^2)(x) \\ &\quad + \delta_{\nu,2}(\theta^{-1})_{21}(x^1 \star f_{pq} + f_{pq} \star x^1)(x) \\ &\quad + \delta_{\nu,3}(\theta^{-1})_{34}(x^4 \star f_{pq} + f_{pq} \star x^4)(x) \\ &\quad + \delta_{\nu,4}(\theta^{-1})_{43}(x^3 \star f_{pq} + f_{pq} \star x^3)(x)) \\ &= \frac{i}{\sqrt{2\theta}} \sum_{p,q \in \mathbb{N}^2} \psi_{pq} \left(-(\delta_{\nu,1} + i\delta_{\nu,2})(\bar{a}^1 \star f_{pq} + f_{pq} \star \bar{a}^1) \right. \\ &\quad + (\delta_{\nu,1} - i\delta_{\nu,2})(a^1 \star f_{pq} + f_{pq} \star a^1) \\ &\quad - (\delta_{\nu,3} + i\delta_{\nu,4})(\bar{a}^2 \star f_{pq} + f_{pq} \star \bar{a}^2) \\ &\quad \left. + (\delta_{\nu,3} - i\delta_{\nu,4})(a^2 \star f_{pq} + f_{pq} \star a^2) \right) (x) \\ &= \frac{i}{\sqrt{2\theta}} \sum_{p,q \in \mathbb{N}^2} \psi_{pq} \\ &\quad \times \left(-(\delta_{\nu,1} + i\delta_{\nu,2}) \left(\sqrt{p^1+1} f_{\frac{p^1+1 q^1}{p^2 q^2}} + \sqrt{q^1} f_{\frac{p^1 q^1-1}{p^2 q^2}} \right) \right. \\ &\quad + (\delta_{\nu,1} - i\delta_{\nu,2}) \left(\sqrt{p^1} f_{\frac{p^1-1 q^1}{p^2 q^2}} + \sqrt{q^1+1} f_{\frac{p^1 q^1+1}{p^2 q^2}} \right) \\ &\quad - (\delta_{\nu,3} + i\delta_{\nu,4}) \left(\sqrt{p^2+1} f_{\frac{p^1 q^1}{p^2+1 q^2}} + \sqrt{q^2} f_{\frac{p^1 q^1}{p^2 q^2-1}} \right) \\ &\quad \left. + (\delta_{\nu,3} - i\delta_{\nu,4}) \left(\sqrt{p^2} f_{\frac{p^1 q^1}{p^2-1 q^2}} + \sqrt{q^2+1} f_{\frac{p^1 q^1}{p^2 q^2+1}} \right) \right) (x) \\ &= \frac{i}{\sqrt{2\theta}} \sum_{p,q \in \mathbb{N}^2} \left(-(\delta_{\nu,1} + i\delta_{\nu,2}) \right. \\ &\quad \times \left(\sqrt{p^1} \psi_{\frac{p^1-1 q^1}{p^2 q^2}} + \sqrt{q^1+1} \psi_{\frac{p^1 q^1+1}{p^2 q^2}} \right) \\ &\quad \left. + (\delta_{\nu,1} - i\delta_{\nu,2}) \left(\sqrt{p^1+1} \psi_{\frac{p^1+1 q^1}{p^2 q^2}} + \sqrt{q^1} \psi_{\frac{p^1 q^1-1}{p^2 q^2}} \right) \right) \end{aligned}$$

$$\begin{aligned}
& -(\delta_{\nu,3} + i\delta_{\nu,4}) \left(\sqrt{p^2} \psi_{p^2-1q^2}^{p^1 q^1} + \sqrt{q^2+1} \psi_{p^2q^2+1}^{p^1 q^1} \right) \\
& + (\delta_{\nu,3} - i\delta_{\nu,4}) \left(\sqrt{p^2+1} \psi_{p^2+1q^2}^{p^1 q^1} + \sqrt{q^2} \psi_{p^2q^2-1}^{p^1 q^1} \right) \Big) f_{pq}(x). \tag{A.3}
\end{aligned}$$

Partial traces with two kernels

We do not imply the double index notation here.

$$\begin{aligned}
& \sum_{n=0}^{\infty} K_{nm;mn}(t') K_{n+1,c;n+1}(t-t') \\
& = \sum_n \sum_{u=0}^{\min(n,m)} \binom{m}{u} \binom{n}{u} e^{2\Omega t'(1+2u)} \left(1 - e^{-4\Omega t'}\right)^{2u} \\
& \quad \times \left(\frac{1-\Omega^2}{4\Omega}\right)^{2u} X_{\Omega}(t')^{n+m+1} \\
& \quad \times \sum_{v=0}^{\min(n+1,c)} \binom{n+1}{v} \binom{c}{v} e^{2\Omega(t-t')(1+2v)} \\
& \quad \times \left(1 - e^{-4\Omega(t-t')}\right)^{2v} \left(\frac{1-\Omega^2}{4\Omega}\right)^{2v} X_{\Omega}(t-t')^{n+c+2} \\
& = X_{\Omega}(t')^{m+1} X_{\Omega}(t-t')^{c+2} \sum_n (X_{\Omega}(t') X_{\Omega}(t-t'))^n \\
& \quad \times \sum_{u=0}^{\min(m,n)} \binom{m}{u} \binom{n}{u} e^{2\Omega t'(1+2u)} \\
& \quad \times \left(1 - e^{-4\Omega t'}\right)^{2u} \left(\frac{1-\Omega^2}{4\Omega}\right)^{2u} \\
& \quad \times \sum_{v=0}^{\min(n+1,c)} \binom{n+1}{v} \binom{c}{v} e^{2\Omega(t-t')(1+2v)} \\
& \quad \times \left(1 - e^{-4\Omega(t-t')}\right)^{2v} \left(\frac{1-\Omega^2}{4\Omega}\right)^{2v} \\
& = \frac{1}{t(1+\Omega^2)} + \mathcal{O}(t^0, t'^0), \tag{A.4}
\end{aligned}$$

and we also need to consider the following partial sum:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sqrt{n+1} K_{nm;m+1,n+1}(t') K_{n+1,c;n+1}(t-t') \\
& = \sum_n \sqrt{n+1} \sum_{u=0}^{\min(n,m)} \sqrt{\binom{n+1}{u+1} \binom{n}{u} \binom{m+1}{u+1} \binom{m}{u}} e^{2\Omega t'(2+2u)} \\
& \quad \times \left(1 - e^{-4\Omega t'}\right)^{2u+1} \left(\frac{1-\Omega^2}{4\Omega}\right)^{2u+1} X_{\Omega}(t')^{n+m+2} \\
& \quad \times \sum_{v=0}^{\min(n+1,c)} \binom{n+1}{v} \binom{c}{v} e^{2\Omega(t-t')(1+2v)} \\
& \quad \times \left(1 - e^{-4\Omega(t-t')}\right)^{2v} \left(\frac{1-\Omega^2}{4\Omega}\right)^{2v} X_{\Omega}(t-t')^{n+c+2}
\end{aligned}$$

$$\begin{aligned}
& = X_{\Omega}(t')^{m+2} X_{\Omega}(t-t')^{c+2} \sqrt{m+1} \sum_n (X_{\Omega}(t') X_{\Omega}(t-t'))^n \\
& \quad \times \sum_{u=0}^{\min(n,m)} \binom{n+1}{u+1} \binom{m}{u} e^{2\Omega t'(2+2u)} \\
& \quad \times \left(1 - e^{-4\Omega t'}\right)^{2u+1} \left(\frac{1-\Omega^2}{4\Omega}\right)^{2u+1} \\
& \quad \times \sum_{v=0}^{\min(n+1,c)} \binom{n+1}{v} \binom{c}{v} e^{2\Omega(t-t')(1+2v)} \\
& \quad \times \left(1 - e^{-4\Omega(t-t')}\right)^{2v} \left(\frac{1-\Omega^2}{4\Omega}\right)^{2v} \\
& = \sqrt{m+1} \frac{1-\Omega^2}{(1+\Omega^2)^2} \frac{t'}{t^2} + \mathcal{O}(t'^0, t^0). \tag{A.5}
\end{aligned}$$

Expressions in the matrix basis

$$\begin{aligned}
& \frac{1}{4\pi^2\theta^2} \int d^4x \, 4 \tilde{x}_{\mu} \star \tilde{x}^{\mu} \star A_{\nu} \star A^{\nu} = \\
& \quad \sum_{p,q \in \mathbb{N}^2} \frac{8}{\theta} (A_{\mu} \star A^{\mu})_{q^1 q^1}^{p^1 p^1} (q^1 + q^2 + 1) \tag{A.6}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4\pi^2\theta^2} \int d^4x \, \{\tilde{x}_{\nu}, A^{\nu}\}_{\star} \star \{\tilde{x}_{\mu}, A^{\mu}\}_{\star} = \\
& \frac{1}{\theta} \left(A_{p^1 q^1}^{(1+)} A_{q^1 p^1}^{(1-)} (p^1 + q^1 + 1) + A_{p^1 q^1}^{(2+)} A_{q^1 p^1}^{(2-)} (p^2 + q^2 + 1) \right. \\
& \quad + \sqrt{p^1} \sqrt{q^1} A_{p^1-1 q^1-1}^{(1+)} A_{q^1 p^1}^{(1-)} + \sqrt{p^1} \sqrt{q^1} A_{p^1 q^1}^{(1+)} A_{q^1-1 p^1-1}^{(1-)} \\
& \quad + \sqrt{p^2} \sqrt{q^2} A_{p^2-1 q^2-1}^{(2+)} A_{q^2 p^2}^{(2-)} + \sqrt{p^2} \sqrt{q^2} A_{p^2 q^2}^{(2+)} A_{q^2-1 p^2-1}^{(2-)} \\
& \quad - \sqrt{p^1} \sqrt{q^1} A_{p^1-1 q^1}^{(1+)} A_{q^1-1 p^1}^{(1-)} - \sqrt{p^1} \sqrt{p^1+1} A_{p^1-1 q^1}^{(1+)} A_{q^1 p^1+1}^{(1-)} \\
& \quad - \sqrt{p^2} \sqrt{q^2} A_{p^2-1 q^2}^{(2+)} A_{q^2-1 p^2}^{(2-)} - \sqrt{p^2} \sqrt{p^2+1} A_{p^2-1 q^2}^{(2+)} A_{q^2 p^2+1}^{(2-)} \\
& \quad - \sqrt{p^1} \sqrt{q^1} A_{p^1 q^1-1}^{(1-)} A_{q^1 p^1-1}^{(1-)} - \sqrt{p^1} \sqrt{p^1+1} A_{p^1+1 q^1}^{(1-)} A_{q^1 p^1-1}^{(1-)} \\
& \quad - \sqrt{p^2} \sqrt{q^2} A_{p^2 q^2-1}^{(2-)} A_{q^2 p^2-1}^{(2-)} - \sqrt{p^2} \sqrt{p^2+1} A_{p^2+1 q^2}^{(2-)} A_{q^2 p^2-1}^{(2-)} \\
& \quad - \sqrt{p^1} \sqrt{q^2} A_{p^1-1 q^1}^{(1+)} A_{q^1 p^1}^{(2+)} - \sqrt{p^1} \sqrt{p^2} A_{p^1-1 q^1}^{(1+)} A_{q^1 p^1}^{(2+)} \\
& \quad - \sqrt{q^1} \sqrt{p^2} A_{p^1 q^1}^{(1+)} A_{q^1-1 p^1}^{(2+)} - \sqrt{q^1} \sqrt{q^2} A_{p^1 q^1}^{(1+)} A_{q^1-1 p^1}^{(2+)} \\
& \quad + \sqrt{p^1} \sqrt{q^2} A_{p^1-1 q^1}^{(1+)} A_{q^1 p^1}^{(2-)} + \sqrt{p^1} \sqrt{p^2} A_{p^1-1 q^1}^{(1+)} A_{q^1 p^1}^{(2-)} \\
& \quad + \sqrt{q^1} \sqrt{q^2} A_{p^1 q^1}^{(1+)} A_{q^1-1 p^1}^{(2-)} + \sqrt{q^1} \sqrt{p^2} A_{p^1 q^1}^{(1+)} A_{q^1-1 p^1}^{(2-)} \Big)
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
& + \sqrt{q^1} \sqrt{p^2} A_{p^1 q^1 p^1}^{(1-)} A_{qp}^{(2+)} + \sqrt{q^1} \sqrt{q^2} A_{p^1 q^1 p^1}^{(1-)} A_{q^1 p^1}^{(2+)} \\
& + \sqrt{p^1} \sqrt{p^2} A_{p^1 q^1}^{(1-)} A_{q^1 p^1}^{(2+)} + \sqrt{p^1} \sqrt{q^2} A_{p^1 q^1}^{(1-)} A_{q^1 p^1}^{(2+)} \\
& - \sqrt{p^1} \sqrt{p^2} A_{pq}^{(1-)} A_{q^1 p^1}^{(2-)} - \sqrt{p^1} \sqrt{q^2} A_{p^1 q^1}^{(1-)} A_{q^1 p^1}^{(2-)} \\
& - \sqrt{q^1} \sqrt{p^2} A_{p^1 q^1}^{(1-)} A_{q^1 p^1}^{(2-)} - \sqrt{q^1} \sqrt{q^2} A_{p^1 q^1}^{(1-)} A_{qp}^{(2-)} \\
& \frac{1}{4\pi^2 \theta^2} \int d^4 x 4 \tilde{x}_\nu \star \tilde{x}_\mu \star A^\nu \star A^\mu = \\
& \frac{2}{\theta} \left(A_{qp}^{(1+)} A_{pq}^{(1-)} (q^1 + p^1 + 1) + A_{qp}^{(2+)} A_{pq}^{(2-)} (q^2 + p^2 + 1) \right. \\
& \left. + \sqrt{q^1 q^2} (A^{(2-)} \star A^{(1+)})_{q^1-1, q^1}^{q^1} \right. \\
& \left. - \sqrt{q^1 q^2} (A^{(2+)} \star A^{(1+)})_{q^1-1, q^1}^{q^1} \right. \\
& \left. + \sqrt{q^1 q^2} (A^{(1+)} \star A^{(2-)})_{q^1-1, q^1}^{q^1} \right. \\
& \left. - \sqrt{q^1 q^2} (A^{(1+)} \star A^{(2+)})_{q^1-1, q^1}^{q^1} \right. \\
& \left. + \sqrt{q^1 q^2} (A^{(2+)} \star A^{(1-)})_{q^1-1, q^1}^{q^1} \right. \\
& \left. - \sqrt{q^1 q^2} (A^{(2-)} \star A^{(1-)})_{q^1, q^1-1}^{q^1} \right. \\
& \left. + \sqrt{q^1 q^2} (A^{(1-)} \star A^{(2+)})_{q^1, q^1-1}^{q^1} \right. \\
& \left. - \sqrt{q^1 q^2} (A^{(1-)} \star A^{(2-)})_{q^1, q^1-1}^{q^1} \right. \\
& \left. - \sqrt{q^1 (q^1 + 1)} (A^{(1-)} \star A^{(1-)})_{q^1+1, q^1-1}^{q^1} \right. \\
& \left. - \sqrt{q^1 (q^1 + 1)} (A^{(1+)} \star A^{(1+)})_{q^1-1, q^1+1}^{q^1} \right. \\
& \left. - \sqrt{q^2 (q^2 + 1)} (A^{(2-)} \star A^{(2-)})_{q^2+1, q^2-1}^{q^1} \right. \\
& \left. - \sqrt{q^2 (q^2 + 1)} (A^{(2+)} \star A^{(2+)})_{q^2-1, q^2+1}^{q^1} \right) \\
& \frac{1}{4\pi^2 \theta^2} \int d^4 x 4 (\tilde{x} \cdot A^\sigma) \star A_\sigma = \\
& \frac{2}{\theta} \left\{ 2 A_{pq}^\sigma (A_\sigma)_{qp} (p^1 + p^2 + 1) \right. \\
& \left. + \sqrt{p^1+1} \sqrt{q^1+1} \left(A_{p^1+1, q^1+1}^{(1+)} A_{qp}^{(1-)} + A_{pq}^{(1+)} A_{q^1+1, p^1+1}^{(1-)} \right) \right. \\
& \left. + \sqrt{p^1+1} \sqrt{q^1+1} \left(A_{p^1+1, q^1+1}^{(2+)} A_{qp}^{(2-)} + A_{pq}^{(2+)} A_{q^1+1, p^1+1}^{(2-)} \right) \right. \\
& \left. + \sqrt{p^2+1} \sqrt{q^2+1} \left(A_{p^1, q^1}^{(1+)} A_{qp}^{(1-)} + A_{pq}^{(1+)} A_{q^1, p^1}^{(1-)} \right) \right. \\
& \left. + \sqrt{p^2+1} \sqrt{q^2+1} \left(A_{p^1, q^1}^{(2+)} A_{qp}^{(2-)} + A_{pq}^{(2+)} A_{q^1, p^1}^{(2-)} \right) \right\} \\
& \frac{1}{4\pi^2 \theta^2} \int d^4 x \tilde{x}_\nu \star A_\mu \star A^\nu \star A^\mu = \\
& \frac{i}{\sqrt{2\theta}} \sum_q \left\{ \sqrt{q^1+1} \left(- (A_\mu \star A^{(1+)} \star A^\mu)_{q^1, q^1} \right. \right. \\
& \left. \left. + (A_\mu \star A^{(1-)} \star A^\mu)_{q^1+1, q^1} \right) \right. \\
& \left. + \sqrt{q^2+1} \left(- (A_\mu \star A^{(2+)} \star A^\mu)_{q^1, q^1} \right. \right. \\
& \left. \left. + (A_\mu \star A^{(2-)} \star A^\mu)_{q^2+1, q^2} \right) \right\}, \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{q^1 q^2} (A^{(2-)} \star A^{(1+)})_{q^1-1, q^1}^{q^1} \\
& - \sqrt{q^1 q^2} (A^{(2+)} \star A^{(1+)})_{q^1-1, q^1}^{q^1} \\
& + \sqrt{q^1 q^2} (A^{(1+)} \star A^{(2-)})_{q^1-1, q^1}^{q^1} \\
& - \sqrt{q^1 q^2} (A^{(1+)} \star A^{(2+)})_{q^1-1, q^1}^{q^1} \\
& + \sqrt{q^1 q^2} (A^{(2+)} \star A^{(1-)})_{q^1-1, q^1}^{q^1} \\
& - \sqrt{q^1 q^2} (A^{(2-)} \star A^{(1-)})_{q^1, q^1-1}^{q^1} \\
& + \sqrt{q^1 q^2} (A^{(1-)} \star A^{(2+)})_{q^1, q^1-1}^{q^1} \\
& - \sqrt{q^1 q^2} (A^{(1-)} \star A^{(2-)})_{q^1, q^1-1}^{q^1} \\
& - \sqrt{q^1 (q^1 + 1)} (A^{(1-)} \star A^{(1-)})_{q^1+1, q^1-1}^{q^1} \\
& - \sqrt{q^1 (q^1 + 1)} (A^{(1+)} \star A^{(1+)})_{q^1-1, q^1+1}^{q^1} \\
& - \sqrt{q^2 (q^2 + 1)} (A^{(2-)} \star A^{(2-)})_{q^2+1, q^2-1}^{q^1} \\
& - \sqrt{q^2 (q^2 + 1)} (A^{(2+)} \star A^{(2+)})_{q^2-1, q^2+1}^{q^1} \\
& \frac{1}{4\pi^2 \theta^2} \int d^4 x 4 (\tilde{x} \cdot A^\sigma) \star A_\sigma = \\
& \frac{2}{\theta} \left\{ 2 A_{pq}^\sigma (A_\sigma)_{qp} (p^1 + p^2 + 1) \right. \\
& \left. + \sqrt{p^1+1} \sqrt{q^1+1} \left(A_{p^1+1, q^1+1}^{(1+)} A_{qp}^{(1-)} + A_{pq}^{(1+)} A_{q^1+1, p^1+1}^{(1-)} \right) \right. \\
& \left. + \sqrt{p^1+1} \sqrt{q^1+1} \left(A_{p^1+1, q^1+1}^{(2+)} A_{qp}^{(2-)} + A_{pq}^{(2+)} A_{q^1+1, p^1+1}^{(2-)} \right) \right. \\
& \left. + \sqrt{p^2+1} \sqrt{q^2+1} \left(A_{p^1, q^1}^{(1+)} A_{qp}^{(1-)} + A_{pq}^{(1+)} A_{q^1, p^1}^{(1-)} \right) \right. \\
& \left. + \sqrt{p^2+1} \sqrt{q^2+1} \left(A_{p^1, q^1}^{(2+)} A_{qp}^{(2-)} + A_{pq}^{(2+)} A_{q^1, p^1}^{(2-)} \right) \right\} \\
& \frac{1}{4\pi^2 \theta^2} \int d^4 x (\tilde{x}_\nu \star A^\nu + A^\nu \star \tilde{x}_\nu) \star A_\mu \star A^\mu = \\
& \frac{i}{\sqrt{2\theta}} \left\{ - \sqrt{p^1} A_{p^1-1, q^1}^{(1+)} (A_\mu \star A^\mu)_{qp} \right. \\
& - \sqrt{q^1+1} A_{p^1, q^1+1}^{(1+)} (A_\mu \star A^\mu)_{qp} \\
& + \sqrt{p^1+1} A_{p^1+1, q^1}^{(1-)} (A_\mu \star A^\mu)_{qp} \\
& + \sqrt{q^1} A_{p^1, q^1-1}^{(1-)} (A_\mu \star A^\mu)_{qp} \\
& - \sqrt{p^2} A_{p^1, q^1}^{(2+)} (A_\mu \star A^\mu)_{qp} \\
& - \sqrt{q^2+1} A_{p^1, q^1}^{(2+)} (A_\mu \star A^\mu)_{qp} \\
& + \sqrt{p^2+1} A_{p^1, q^1}^{(2-)} (A_\mu \star A^\mu)_{qp} \\
& \left. + \sqrt{q^2} A_{p^1, q^1}^{(2-)} (A_\mu \star A^\mu)_{qp} \right\} \tag{A.9}
\end{aligned}$$

where e.g.,

$$\begin{aligned}
& (A_\mu \star A^{(1+)} \star A^\mu)_{q^1, q^1}^{q^1} = \\
& \frac{1}{2} \left(A_{qa}^{(1+)} A_{ab}^{(1+)} A_{b^1, q^1+1}^{(1-)} + A_{qa}^{(1-)} A_{ab}^{(1+)} A_{b^1, q^1+1}^{(1+)} \right. \\
& \left. + A_{qa}^{(2+)} A_{ab}^{(1+)} A_{b^1, q^1+1}^{(2-)} + A_{qa}^{(2-)} A_{ab}^{(1+)} A_{b^1, q^1+1}^{(2+)} \right). \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4\pi^2 \theta^2} \int d^4 x (\tilde{x}_\nu \star A^\nu + A^\nu \star \tilde{x}_\nu) \star A_\mu \star A^\mu = \\
& \frac{i}{\sqrt{2\theta}} \left\{ - \sqrt{p^1} A_{p^1-1, q^1}^{(1+)} (A_\mu \star A^\mu)_{qp} \right. \\
& - \sqrt{q^1+1} A_{p^1, q^1+1}^{(1+)} (A_\mu \star A^\mu)_{qp} \\
& + \sqrt{p^1+1} A_{p^1+1, q^1}^{(1-)} (A_\mu \star A^\mu)_{qp} \\
& + \sqrt{q^1} A_{p^1, q^1-1}^{(1-)} (A_\mu \star A^\mu)_{qp} \\
& - \sqrt{p^2} A_{p^1, q^1}^{(2+)} (A_\mu \star A^\mu)_{qp} \\
& - \sqrt{q^2+1} A_{p^1, q^1}^{(2+)} (A_\mu \star A^\mu)_{qp} \\
& + \sqrt{p^2+1} A_{p^1, q^1}^{(2-)} (A_\mu \star A^\mu)_{qp} \\
& \left. + \sqrt{q^2} A_{p^1, q^1}^{(2-)} (A_\mu \star A^\mu)_{qp} \right\} \tag{A.11}
\end{aligned}$$

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